Exact WKB Analysis of Difference Equations for Bessel Functions

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(Received April 12, 2022)

In this paper the exact WKB analysis, that is, the WKB analysis based on the Borel resummation method, of difference equations for Bessel functions is considered. We investigate their WKB solutions, Stokes curves, and connection formulas by using integral representations of solutions. In particular, we will show that infinitely many new Stokes curves appear for the difference equations for Bessel functions.

Key words : exact WKB analysis, difference equation, Bessel function, new Stokes curve

1. Introduction

WKB solutions, which are divergent solutions of the form of power series in the Planck constant $\hbar$, have been used to solve eigenvalue problems for Schrödinger equations. Since the pioneering work of Voros[1] the Borel resummation method is employed to give analytical meaning to WKB solutions and such analysis based on the Borel resummation is now called the exact WKB analysis. At present, it is applied not only to Schrödinger equations but also to high-order ordinary differential equations and nonlinear equations[2,3]. However, the exact WKB analysis for difference equations is still almost untouched. In this paper, as a first step toward the extension of the exact WKB analysis to difference equations, we consider a difference equation

$$\sigma_\gamma^2\psi - \frac{2\gamma}{x}\sigma_\gamma^1\psi + \psi = 0, \tag{B}$$

where γ and x are variables and σ_γ denotes a small shift operator $\gamma \mapsto \gamma + \hbar$ for γ, that the Bessel functions satisfy from the viewpoint of the exact WKB anaysis. To be more specific, we discuss WKB solutions, Stokes curves and connection formulas for Eq. (B). In particular, we will show that infinitely many new Stokes curves appear for Eq. (B) by using integral representations of Bessel functions.

The plan of this paper is as follows: We first review the basic theory of the exact WKB analysis for Schrödinger equations in Section 2. Then, with the help of contiguity relations, we derive the difference equation Eq. (B) for Bessel functions and discuss its WKB solutions and Stokes curves in Section 3. Sections 4 and 5 are the main part of this paper: Applying the so-called steepest descent method to integral representations of Bessel functions, we investigate the exact WKB theoretic structure of Eq. (B) in details and show that infinitely many new Stokes curves appear for Eq. (B).

2. Brief Review of the Exact WKB Analysis

Following Ref.[2], we briefly review the exact WKB analysis for the Schrödinger equation with a large parameter $\eta = \hbar^{-1}$:

$$\left(-\frac{d^2}{dx^2} + \eta^2 Q(x)\right)\psi = 0, \tag{1}$$

where $Q(x)$ is a polynomial or a rational function. In what follows we assume $\eta > 0$.

A formal solution of Eq. (1) of the form

$$\psi_\pm = \exp(\pm y_0(x)\eta)\sum_{n=0}^{\infty}\psi_{\pm,n}(x)\eta^{-n-1/2} \tag{2}$$

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is called a WKB solution. Here $y_0(x) = \int^x \sqrt{Q(x)}dx$ and $\psi_{\pm,n}(x)$ $(n \geq 0)$ are recursively determined. For the construction of WKB solutions see Ref.[2]. As WKB solutions are divergent in almost all cases, we employ the following Borel resummation method in the exact WKB analysis.

Definition 1. Let $\alpha \notin \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. For a formal series of the form

$$\psi(\eta) = \exp(y_0\eta) \sum_{n=0}^{\infty} \psi_n \eta^{-n-\alpha} \tag{3}$$

where y_0, ψ_n are independent of η, we define its Borel transform $\psi_B(y)$ and Borel sum $\Psi(\eta)$ by

$$\psi_B(y) = \sum_{n=0}^{\infty} \frac{\psi_n}{\Gamma(n+\alpha)} (y+y_0)^{n+\alpha-1} \tag{4}$$

and

$$\Psi(\eta) = \int_{-y_0}^{\infty} \exp(-y\eta)\psi_B(y)dy, \tag{5}$$

respectively. Here $\Gamma(z)$ denotes the gamma function and the path of the integral (5) is taken to be a straight line emanating from $-y_0$ and being parallel to the positive real axis.

When the Borel sum (5) is well-defined, that is, when the Borel transform $\psi_B(y)$ is convergent and analytically continuable along the integral path of (5), and further the integral (5) converges, we say that a formal series $\psi(\eta)$ is Borel summable.

To describe where WKB solutions are Borel summable, we introduce the following terminology.

Definition 2. (i) A zero of $Q(x)$ is called a turning point of Eq. (1). In particular, a simple zero of $Q(x)$ is called a simple turning point.
(ii) A real one-dimensional curve defined by

$$\Im \int_a^x \sqrt{Q(x)}dx = 0, \tag{6}$$

where a is a turning point, is called a Stokes curve of Eq. (1). A region surrounded by Stokes curves is called a Stokes region.

For example, when $Q(x) = x$, the origin $x = 0$ is a unique turning point and the Stokes curves consist of three straight lines emanating from the origin.

In the case of Eq. (1) the following holds[2].

Theorem 1. *Assume that all turning points of Eq.* (1) *are simple and that none of the turning points are connected by Stokes curves. Then we have*
(i) *Except on Stokes curves WKB solutions are Borel summable and define analytic solutions.*
(ii) *On each Stokes curve Stokes phenomena occur with Borel sums of WKB solutions. That is, after crossing a Stokes curve the analytic continuation of Borel sums $\Psi_\pm$ of WKB solutions become*

$$\Psi_+ \to \tilde{\Psi}_+ \pm C\tilde{\Psi}_-, \quad \Psi_- \to \tilde{\Psi}_-, \tag{7}$$

or

$$\Psi_+ \to \tilde{\Psi}_+, \quad \Psi_- \to \tilde{\Psi}_- \pm C\tilde{\Psi}_+, \tag{8}$$

where C is a constant and $\tilde{\Psi}_\pm$ are Borel sums of WKB solutions in a new Stokes region adjacent to the original region across the Stokes curve.

The relations (7)-(8) are called the connection formulas for WKB solutions. For the determination of the constant C see Ref.[2].

The purpose of this paper is to investigate to what extent these results can be extended to the difference equation (B) for Bessel functions.

3. Bessel's Difference Equation — Its WKB Solutions and Stokes Geometry

In this section we recall some properties of Bessel functions and introduce the difference equation that the Bessel functions satisfy.

Definition 3. The Bessel function $J_a(x)$ is defined by

$$J_a(x) = \left(\frac{x}{2}\right)^a \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{\Gamma(n+a+1)\, n!}, \tag{9}$$

where a is a complex parameter.

As is well-known, $J_a(x)$ and $J_{-a}(x)$ satisfy

$$x^2\frac{d^2\psi}{dx^2}+x\frac{d\psi}{dx}+(x^2-a^2)\psi=0. \tag{10}$$

Furthermore, the Bessel functions satisfy the following relation (12) known as the contiguity relation or recurrence formula[4,5]:

Proposition 2. *Let*

$$H(a)=\frac{\partial}{\partial x}-\frac{a}{x},\ \ B(a)=\frac{\partial}{\partial x}+\frac{a}{x}. \tag{11}$$

Then the following relation holds.

$$H(a)J_a(x)=-J_{a+1}(x),\ \ B(a)J_a(x)=J_{a-1}(x). \tag{12}$$

In other words, $H(a)$ is a raising operator that changes the parameter a to $a+1$ and $B(a)$ is a lowering operator that changes a to $a-1$.

By Proposition 2, letting $\sigma_a^1\psi := -H(a)\psi = -\psi'+(a/x)\psi$, we find that σ_a^1 acts on the Bessel functions $J_a(x)$ as a shift operator that increases the parameter a by 1. Rewriting Bessel's differential equation (10) by using the shift operator σ_a^1, we thus obtain the following difference equation for Bessel functions $J_a(x)$.

$$\sigma_a^2\psi-\frac{2(a+1)}{x}\sigma_a^1\psi+\psi=0. \tag{13}$$

In what follows we study the difference equation (13) from the viewpoint of the exact WKB analysis. For that purpose we introduce a large parameter η into (13) in the following way:

$$x\mapsto \eta x,\quad a+1\mapsto \eta\gamma. \tag{14}$$

We also let σ_γ^1 be a small shift operator for the paramater γ: $\sigma_\gamma^1 f(\gamma)=f(\gamma+\eta^{-1})$. Then Eqs. (10) and (13) become

$$x^2\frac{d^2\psi}{dx^2}+x\frac{d\psi}{dx}+\eta^2(x^2-(\gamma-\eta^{-1})^2)\psi=0 \tag{15}$$

and

$$\sigma_\gamma^2\psi-\frac{2\gamma}{x}\sigma_\gamma^1\psi+\psi=0, \tag{B}$$

respectively. In particular, Eq. (B) is the equation that has already appeared in Introduction. Eq. (B) is the main object of our study.

We first construct WKB solutions of Eq. (B). In what follows we consider an unknown function $\psi(\gamma)$ to be a function of γ (while x is considered to be just a parameter) and assume that ψ has the following expansion:

$$\psi=\exp\left(\eta\int^\gamma\phi(\gamma)d\gamma\right)\sum_{n=0}^\infty\psi_n(\gamma)\eta^{-n-1/2}. \tag{16}$$

Further, via the Taylor expansion, we regard the small shift operator σ_γ^1 as an infinite order differential operator

$$\sigma_\gamma^1\psi(\gamma)=\psi(\gamma+\eta^{-1})=\sum_{n=0}^\infty\frac{\eta^{-n}}{n!}\frac{d^n}{d\gamma^n}\psi(\gamma). \tag{17}$$

Then, substituting (16) into Eq. (B) with applying (17) to (16) and comparing like powers with respect to η, we find

$$\left(e^{2\phi}-\frac{2\gamma}{x}e^\phi+1\right)\psi_0=0, \tag{18}$$

and

$$\left(2e^\phi-\frac{\gamma}{x}\right)\phi'\psi_n+2\left(e^\phi-\frac{\gamma}{x}\right)\psi_n'=G_n, \tag{19}$$

where $n=0,1,2,\ldots$ and G_n is a function of ϕ, $\psi_0,\ldots,$ and ψ_{n-1}. Hence,

$$\phi=\phi_\pm(\gamma)=\log\left(\frac{\gamma}{x}\pm\sqrt{\left(\frac{\gamma}{x}\right)^2-1}\right) \tag{20}$$

and $\phi_n(\gamma)$ $(n=0,1,2,\ldots)$ are recursively determined by (19). In this way we obtain WKB solutions of (B).

Remark 1. Corresponding to the choice of $\phi_\pm(\gamma)$, there are two WKB solutions $\psi_\pm(\gamma)$. More precisely, due to the multivaluedness of the logarithmic function, there exist infinitely many WKB solutions $\psi_{\pm,m}(\gamma)$ $(m\in\mathbb{Z})$ corresponding to

$$\phi_{\pm,m}(\gamma)=\mathrm{Log}\left(\frac{\gamma}{x}\pm\sqrt{\left(\frac{\gamma}{x}\right)^2-1}\right)+2m\pi i,$$

where Log denotes the principal value.

Next we define turning points and Stokes curves of the difference equation (B). Taking Definition 2 into account, we define them as follows:

Definition 4. (i) A point γ satisfying $\phi_+(\gamma) = \phi_-(\gamma)$ is called a turning point of Eq. (B). Thus $\gamma = \pm x$ are the turning points of Eq. (B).
(ii) A Stokes curve of Eq. (B) is defined by

$$\Im \int_{\gamma_0}^{\gamma} (\phi_+(\gamma) - \phi_-(\gamma))\, d\gamma = 0, \tag{21}$$

where γ_0 $(= \pm x)$ is a turning point.

Fig. 1 shows the configuration of turning points (denoted by $\triangle$) and Stokes curves for $x = 1$. Different from the case of Schrödinger equations, we can observe that Stokes curves do really cross in the case of Eq. (B). Such crossing of Stokes curves are common features for higher order differential equations[3]. In fact, the small shift operator is here regarded as an infinite order differential operator. This problem will be discussed in more details in Section 5.

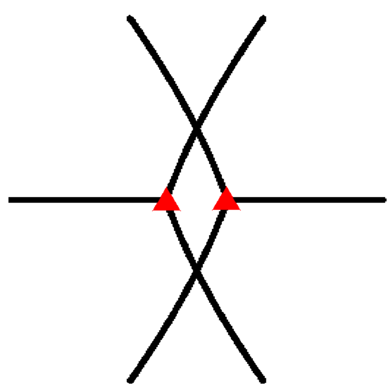

Fig. 1. Stokes curves for $x = 1$

4. Exact WKB Analysis of Bessel's Difference Equation via Integral Representations

It is well-known that solutions of Eq. (10) admit the following integral representation[4,5]:

$$\int_\Gamma \exp\left[\frac{x}{2}\Big(u - \frac{1}{u}\Big)\right] u^{-a-1} du, \tag{22}$$

where Γ is a suitably chosen integral path connecting 0 or ∞ so that the integral converges. After the introduction of a large parameter η by (14), this becomes

$$\int_\Gamma \exp\left[\frac{\eta x}{2}\Big(u - \frac{1}{u}\Big)\right] u^{-\eta\gamma} du = \int_\Gamma e^{\eta f(\gamma,u)} du \tag{23}$$

with

$$f(\gamma, u) = \frac{x}{2}\Big(u - \frac{1}{u}\Big) - \gamma \log u. \tag{24}$$

Then we can confirm the following

Proposition 3. *The integral representation* (23) *also satisfies the difference equation* (B)*.*

Proof. Substitution of (23) into (B) entails

$$\begin{aligned}&\int_\Gamma \exp\left[\frac{\eta x}{2}\Big(u - \frac{1}{u}\Big)\right] u^{-\eta\gamma}\Big(\frac{1}{u^2} - \frac{2\gamma}{x}\frac{1}{u} + 1\Big) du \\ &= \frac{2}{\eta x}\int_\Gamma \frac{\partial}{\partial u}\Big(\exp\left[\frac{\eta x}{2}\Big(u - \frac{1}{u}\Big)\right] u^{-\eta\gamma}\Big) du = 0.\end{aligned}$$

This completes the proof.

Making use of the integral representation (23) and following Ref.[6], we discuss the Borel summability and Stokes phenomena for WKB solutions of Eq. (B) in this section.

To this end, we apply the so-called "steepest descent method" to (23), that is, we consider the integral (23) along a steepest descent path of $\Re f$ through a saddle point of f to study the asymptotic behavior of the integrals (23).

Definition 5. (i) A point satisfying $\partial f/\partial u = 0$ is called a saddle point of f.
(ii) A steepest descent path of $\Re f$ is an integral curve of $-\mathrm{grad}\,\Re f = -\big(\partial(\Re f)/\partial v, \partial(\Re f)/\partial w\big)$ (where $u = v + iw$), that is, a curve on which $\Re f$ decreases most rapidly.

Note that, since $f(\gamma, u)$ is a holomorphic function, $\Im f$ is constant on each steepest descent path of $\Re f$.

In our case the saddle points of (23) are explicitly given by

$$u_\pm := \frac{\gamma}{x} \mp \sqrt{\Big(\frac{\gamma}{x}\Big)^2 - 1}. \tag{25}$$

(Here, for the sake of later convenience, we use the opposite sign $\mp$ in the right-hand side.) Let $\Gamma_\pm$ be the steepest descent path of $\Re f$ passing through the saddle point $u = u_\pm$ and consider the integral

$$\Psi_\pm^{\mathrm{IR}} := \int_{\Gamma_\pm} e^{\eta f(\gamma,u)} du. \tag{26}$$

As $\Gamma_\pm$ extends in two directions from $u_\pm$, we denote these two portions of $\Gamma_\pm$ by $\Gamma_{\pm,1}$ and $\Gamma_{\pm,2}$. Decomposing the integral (26) into these two portions, we employ a change of integral variable $y = -(f(\gamma,u) - f(\gamma,u_\pm))$ for each integral. If we assume that the steepest descent path $\Gamma_\pm$ does not hit any other saddle point, the new integral variable y increases monotonically from 0 to ∞ on $\Gamma_{\pm,j}$ $(j = 1,2)$ and we obtain

$$\begin{aligned}\Psi_\pm^{\mathrm{IR}} =& e^{\eta f(\gamma,u_\pm)} \int_0^\infty e^{-\eta y} \\ & \left(\frac{1}{f_u}\bigg|_{u=u_{\pm,1}(\gamma,y)} - \frac{1}{f_u}\bigg|_{u=u_{\pm,2}(\gamma,y)} \right) dy,\end{aligned} \tag{27}$$

where $f_u = \partial f/\partial u$ and $u_{\pm,j}$ $(j = 1,2)$ is the inverse function of $y = -(f(\gamma,u) - f(\gamma,u_\pm))$ on $\Gamma_{\pm,j}$.

For the exponential term $f(\gamma,u_\pm)$ on the right-hand side of (27), the following holds:

Proposition 4. $f(\gamma,u_\pm)$ *can be expressed as*

$$f(\gamma,u_\pm) = \int^\gamma \phi_\pm(\gamma) d\gamma. \tag{28}$$

Proof. Since $(\partial f/\partial u)(\gamma,u_\pm) = 0$, we have

$$\frac{\partial}{\partial\gamma} f(\gamma,u_\pm) = \frac{\partial f}{\partial\gamma}(\gamma,u_\pm) = \log(u_\pm)^{-1}.$$

As $u_+u_- = 1$ holds, we obtain

$$\frac{\partial}{\partial\gamma} f(\gamma,u_\pm) = \log u_\mp = \phi_\pm(\gamma).$$

This completes the proof.

Furthermore, to the integral on the right-hand side of (27) we apply Watson's lemma[7]:

Proposition 5. (Watson's lemma) *Let $f(t)$ be an analytic function on $\{0 < |t| < R+\delta, |\arg t| < \Delta\}$ for some $R,\delta,\Delta > 0$. Furthermore, for some $q,K,b > 0$ we assume the following:*
(i) *Near $t = 0$ $f(t)$ has a convergent expansion $f(t) = \sum_{k\geq 1} a_k t^{k/q-1}$ $(0 < |t| \leq R)$.*
(ii) *$|f(r)| \leq K e^{br}$ holds for $r \geq R$.*
Then, the asymptonic expansion formula

$$\int_0^\infty e^{-zt} f(t) dt \sim \sum_{k=1}^\infty a_k \Gamma\Big(\frac{k}{q}\Big) z^{-k/q}$$

holds for $z \to \infty$, $|\arg z| < \pi/2 - \epsilon$ with $\epsilon > 0$.

We set

$$h(y) := \frac{1}{f_u}\bigg|_{u=u_{\pm,1}(\gamma,y)} - \frac{1}{f_u}\bigg|_{u=u_{\pm,2}(\gamma,y)}. \tag{29}$$

Note that $u = u_\pm$ given by (25) is a nondegenerate saddle point when γ is not a turning point. Therefore, the inverse function $u_{\pm,j}$ $(j = 1,2)$ has the Puiseux expansion of the form

$$u_{\pm,j} = u_\pm + (-1)^j a_{1/2} y^{1/2} + a_1 y + (-1)^j a_{3/2} y^{3/2} + \cdots$$

around $y = 0$ corresponding to $u = u_\pm$. In particular, the coefficients of $u_{\pm,j}$ for a half-integer power of y have the opposite sign. Consequently, we find $h(y)$ has the expansion of the form

$$h(y) = \sum_{n=0}^\infty b_n y^{(2n+1)/2-1} \tag{30}$$

around $y = 0$. Then, by applying Proposition 5 to (27) with $q = 2$, we obtain

$$\Psi_\pm^{\mathrm{IR}} \sim e^{\eta f(\gamma,u_\pm)} \sum_{n=0}^\infty c_n \eta^{-n-1/2} \tag{31}$$

with

$$c_n = b_n \Gamma(n + 1/2). \tag{32}$$

The right-hand side of (31) has exactly the form of WKB solutions (16) in view of Proposition 4. Furthermore, Proposition 3 implies it is a formal solution of (B). Thus the right-hand side of (31), which is denoted by $\psi_\pm^{\mathrm{IR}}$ hereafter, defines a WKB solution of (B). In what follows we discuss this WKB solution $\psi_{\pm}^{\rm IR}$.

It follows from (30) and (32) that the Borel transform of $\psi_{\pm}^{\rm IR}$ coincides with $h(y+f(\gamma,u_{\pm}))$. Hence the Borel sum of $\psi_{\pm}^{\rm IR}$ is

$$\begin{aligned}\int_{-f(\gamma,u_{\pm})}^{\infty} & e^{-\eta y}h(y+f(\gamma,u_{\pm}))dy \\ &= e^{\eta f(\gamma,u_{\pm})}\int_0^{\infty} e^{-\eta y}h(y)dy,\end{aligned}$$

which is nothing but $\Psi_{\pm}^{\rm IR}$. That is, $\Psi_{\pm}^{\rm IR}$ gives the Borel sum of $\psi_{\pm}^{\rm IR}$. Since $h(y)$, that is, the Borel transform of $\psi_{\pm}^{\rm IR}$, is singular only at the saddle points in view of the expression (29), we consequently obtain the following

Theorem 6. (i) *The WKB solution $\psi_{\pm}^{\rm IR}$ of* (B) *is Borel summable if and only if the steepest descent path $\Gamma_{\pm}$ through the saddle point $u=u_{\pm}$ does not hit any other saddle points.*
(ii) *When $\Gamma_{\pm}$ hits another saddle point, a Stokes phenomenon occurs with $\psi_{\pm}^{\rm IR}$.*

Theorem 6 tells us that we can check the Borel summability of the WKB solution $\psi_{\pm}^{\rm IR}$ of (B) by tracing the configuration of steepest descent paths of the integral representation (23). In what follows, using a computer, we will investigate the configuration of steepest descent paths of (23) to examine whether a Stokes phenomenon occurs or not on Stokes curves of Bessel's difference equation (B).

For the sake of simplicity we assume $x=1$ and investigate the configuration of steepest descent paths of (23) around the crossing point, denoted by γ_*, in the upper half plane in Fig. 1. First, the configuration of steepest descent paths at $\gamma=\gamma_*$ is shown in Fig. 2. In Fig. 2 (and also in the following Fig. 5) the saddle points $u=u_{\pm}$ and the singular points $u=0$ of $f(\gamma,u)$ are designated by (red) $\triangle$ and by (blue) $\bullet$, respectively. In particular, the upper saddle point is u_+ and the lower saddle point is u_- in Fig. 2, and the steepest

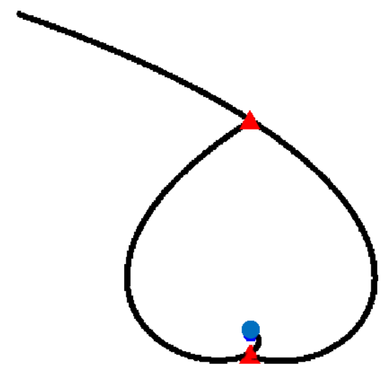

Fig. 2. Steepest descent paths at $\gamma=\gamma_*$

descent path that extends upward (resp., downward) from $u_{\pm}$ is denoted by $\Gamma_{\pm,1}$ (resp., $\Gamma_{\pm,2}$). As is clearly shown in Fig. 2, $\Gamma_{+,2}$ hits u_- and $\Gamma_{-,2}$ hits u_+ simultaneously at $\gamma=\gamma_*$. Thus, according to Theorem 6, the Borel summability of $\psi_{\pm}^{\rm IR}$ breaks down at γ_*. Next, we take sample points $\gamma_1,\gamma_2,\ldots,\gamma_7$ around γ_* (cf. Fig. 3) and investigate the configuration of steepest descent paths at these sample points. The results are

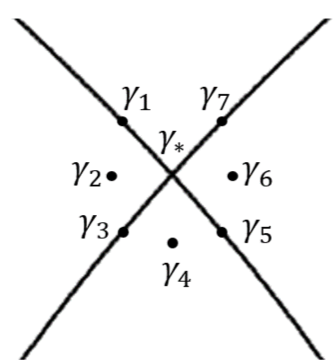

Fig. 3. Sample points $\gamma_1,\gamma_2,\ldots,\gamma_7$ around γ_*

shown in Fig. 5 (a)–(g) placed at the end of the paper. From these figures we can observe that a steepest descent path passing through a saddle point hits another saddle point at points on Stokes curves such as γ_1, γ_3, γ_5 and γ_7, while there exists no such steepest descent paths at points outside Stokes curves. For example, let γ vary from γ_3, γ_4, …, to γ_6. At γ_3 on a Stokes curve emanating from $\gamma=-x=-1$, a steepest descent path $\Gamma_{-,2}$ passing through a lower saddle point u_- hits the other saddle point u_+ (cf. Fig.

5 (c)). This steepest descent path $\Gamma_{-,2}$ goes to infinity without hitting u_+ at γ_4 (cf. Fig. 5 (d)). Then at γ_5, that is, at a point on a Stokes curve emanating from $\gamma = 1$, another steepest descent path $\Gamma_{+,2}$ passing through u_+ hits u_- instead (cf. Fig. 5 (e)). It is also observed that the steepest descent path $\Gamma_{+,2}$ no longer hits u_- at γ_6 (cf. Fig. 5 (f)). In this way, by tracing the configuration of steepest descent paths and combining the results with Theorem 6, we can confirm that the Borel summability of the WKB solution $\psi_\pm^{\mathrm{IR}}$ breaks down on Stokes curves.

Furthermore, by comparing Fig. 5 (d) and Fig. 5 (f), we find that the Borel sum Ψ_+^{IR} near $\gamma = \gamma_4$ is analytically continued to $\Psi_+^{\mathrm{IR}} + \Psi_-^{\mathrm{IR}}$ near $\gamma = \gamma_6$. As a matter of fact, the integral path for Ψ_+^{IR} at $\gamma = \gamma_4$ is the steepest descent path Γ_+, that is, a path that starts from infinity, passes through a saddle point u_+, and goes to the singular point $u = 0$ with passing near another saddle point u_-. This path is expressed as the sum of two steepest descent paths $\Gamma_+ \cup \Gamma_-$ at $\gamma = \gamma_6$ (provided that the orientation of $\Gamma_\pm$ is appropriately chosen). This implies the above formula.

As this change of the configuration of steepest descent paths occurs on every Stokes curve, we thus obtain the following

Theorem 7. *On each Stokes curve of Bessel's difference equation* (B) *Stokes phenomena occur with the Borel sums of the WKB solutions $\psi_\pm^{\mathrm{IR}}$. That is, after crossing a Stokes curve the analytic continuation of the Borel sums $\Psi_\pm^{\mathrm{IR}}$ become*

$$\Psi_+^{\mathrm{IR}} \to \tilde{\Psi}_+^{\mathrm{IR}} + C\tilde{\Psi}_-^{\mathrm{IR}}, \quad \Psi_-^{\mathrm{IR}} \to \tilde{\Psi}_-^{\mathrm{IR}}, \qquad (33)$$

or

$$\Psi_+^{\mathrm{IR}} \to \tilde{\Psi}_+^{\mathrm{IR}}, \quad \Psi_-^{\mathrm{IR}} \to \tilde{\Psi}_-^{\mathrm{IR}} + C\tilde{\Psi}_+^{\mathrm{IR}}, \qquad (34)$$

where $\tilde{\Psi}_\pm^{\mathrm{IR}}$ are Borel sums of $\psi_\pm^{\mathrm{IR}}$ in a new Stokes region adjacent to the original region across the Stokes curve. The constant C equals $+1$ or -1, which depends on the orientation of the steepest descent paths $\Gamma_\pm$.

The relations (33)-(34) are the connection formulas for the WKB solutions $\psi_\pm^{\mathrm{IR}}$ of Bessel's difference equation (B).

5. New Stokes Curves of Bessel's Difference Equation

Berk-Nevins-Roberts[8)] pointed out that Stokes curves may cross for higher order differential equations and that a new Stokes curve may appear from such crossing points of Stokes curves. Here a new Stokes curve means a curve where Stokes phenomena occur with WKB solutions but which does not emanate from a turning point. As we have observed in Section 3, Stokes curves do cross also for Bessel's difference equation (B). In this section we examine whether new Stokes curves appear or not for (B) by using the integral representation (23).

We investigate the configuration of steepest descent paths of (23) at several sample points $\gamma_8, \gamma_9, \ldots, \gamma_{14}$ between γ_7 and γ_1 around the crossing point γ_* (cf. Fig. 4). See Fig. 5 (h)–(n)

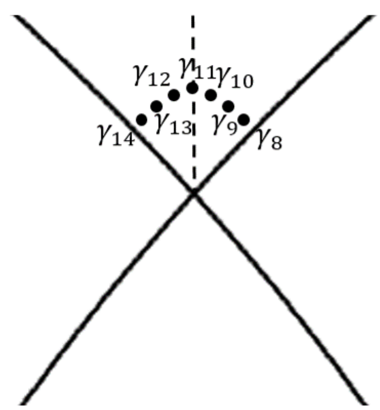

Fig. 4. Sample points $\gamma_8, \gamma_9, \ldots, \gamma_{14}$ between γ_7 and γ_1

at the end of the paper for the results. We can observe the following configuration from these figures (although it is not so clearly visualized). First, at γ_8 neither of $\Gamma_\pm$ hits another saddle point. However, at γ_9 the steepest descent path Γ_- (more precisely, $\Gamma_{-,2}$) passing through u_- hits the other saddle point u_+ after going around the singular point $u = 0$ once (with passing just below u_-). Furthermore, at γ_{10} the steepest descent path Γ_- hits u_+ after going around $u = 0$ twice. It is naturally expected that, when γ varies further from γ_{10} to γ_{11}, Γ_- hits u_+ after going around $u = 0$ three times, four times, ... (with the rotation number increasing) at some points between γ_{10} to γ_{11}. Then, at γ_{11} on the imaginary axis, both the two steepest descent paths $\Gamma_\pm$ hit their starting saddle points $u_\pm$ simultaneously. In the left-half plane similar configurations can be observed, but this time with Γ_+ instead of Γ_-. That is, the steepest descent path Γ_+ passing through u_+ hits u_- after going around $u = 0$ ..., twice, and once at ..., γ_{12}, and γ_{13}.

The above observation indicates that there exist several points outside Stokes curves where a steepest descent path passing through a saddle point hits another saddle point. Theorem 6 implies Stokes phenomena occur with the WKB solutions $\psi_\pm^{\mathrm{IR}}$ at these points and hence these points are considered to be on new Stokes curves passing through the crossing point γ_* of the Stokes curves of Bessel's difference equation (B). As a matter of fact, using the notation explained in Remark 1, the Stokes curve emanating from $\gamma = 1$ of (B) is expressed as

$$\Im \int_1^\gamma (\phi_{+,m} - \phi_{-,m}) d\gamma = 0 \ \ (m \in \mathbb{Z}).$$

Following the convention for the types of Stokes curves used in Ref.[3], this is a Stokes curve of type $(+, m) > (-, m)$. On the other hand, the Stokes curve emanating from $\gamma = -1$ is expressed as

$$\Im \int_{-1}^\gamma (\phi_{+,m} - \phi_{-,m-1}) d\gamma = 0 \ \ (m \in \mathbb{Z})$$

and this is a Stokes curve of type $(-, m-1) > (+, m)$. (Note that it is equivalently said to be of type $(-, m) > (+, m+1)$.) According to the general rule for the types of a new Stokes curve for higher order differential equations[3], it is expected that a new Stokes curve of type $(-, m-1) > (-, m)$ and one of type $(+, m) > (+, m+1)$ appear from the crossing point γ_*. These new Stokes curves are expressed as

$$\Im \int_{\gamma_*}^\gamma (\phi_{-,m} - \phi_{-,m-1}) d\gamma = \Im \int_{\gamma_*}^\gamma (2\pi i) d\gamma = 0$$

and

$$\Im \int_{\gamma_*}^\gamma (\phi_{+,m+1} - \phi_{+,m}) d\gamma = \Im \int_{\gamma_*}^\gamma (2\pi i) d\gamma = 0,$$

respectively. This is nothing but the imaginary axis and this result is consistent with Fig. 5 (k). Furthermore, the general rule also suggests that new Stokes curves of type $(-, m) > (+, m+2)$, $(-, m) > (+, m+3)$, ... and those of type $(+, m) > (-, m+1)$, $(+, m) > (-, m+2)$, ... appear from γ_* as well. The points γ_9, γ_{10}, ..., γ_{12}, γ_{13} where the steepest descent path $\Gamma_\pm$ hits $u_\mp$ after going around $u = 0$ several times are considered to correspond to these new Stokes curves. To be more specific, γ_9 is considered to be on a new Stokes curve of type $(-, m) > (+, m+2)$, γ_{10} on one of type $(-, m) > (+, m+3)$, ..., γ_{12} on one of type $(+, m) > (-, m+2)$, and γ_{13} on one of type $(+, m) > (-, m+1)$. Thus it is confirmed that there exist infinitely many new Stokes curves passing through the crossing point γ_* of Stokes curves for Bessel's difference equation (B).

Remark 2. The new Stokes curves of type $(-, m) > (+, m+\mu)$ and of type $(+, m) > (-, m+\mu)$ ($\mu \in \{1, 2, \ldots\}$) passing through γ_* are defined respectively by the following relations:

$$\Im \int_{\gamma_*}^\gamma \Big[\mathrm{Log}\Big(\frac{\gamma}{x} + \sqrt{\Big(\frac{\gamma}{x}\Big)^2 - 1}\Big) - \mathrm{Log}\Big(\frac{\gamma}{x} - \sqrt{\Big(\frac{\gamma}{x}\Big)^2 - 1}\Big) - 2\pi i\mu\Big] d\gamma = 0, \tag{35}$$

$$\Im \int_{\gamma_*}^\gamma \Big[\mathrm{Log}\Big(\frac{\gamma}{x} - \sqrt{\Big(\frac{\gamma}{x}\Big)^2 - 1}\Big)$$

$$- \operatorname{Log}\Big(\frac{\gamma}{x} + \sqrt{\Big(\frac{\gamma}{x}\Big)^2 - 1}\Big) - 2\pi i\mu\Big] d\gamma = 0. \tag{36}$$

Finally, we consider the connection formulas on these new Stokes curves. Taking the multi-valuedness of $f(\gamma, u)$ defined by (24) into account (here we place a cut on the negative real axis in u-plane to fix the branch of $f(\gamma, u)$), we obtain the following Theorem 8 by the argument similar to that employed in verifying Theorem 7, that is, by tracing the change of the configuration of steepest descent paths that occurs when crossing a new Stokes curve.

Theorem 8. *On a new Stokes curve of type* $(-, m) > (+, m+\mu)$ *and of type* $(+, m) > (-, m+\mu)$ $(\mu \in \{1, 2, \dots\})$ *passing through the crossing point* γ_* *of Stokes curves of Eq.* (B)*, the connection formulas*

$$\Psi_+^{\mathrm{IR}} \to \tilde{\Psi}_+^{\mathrm{IR}}, \quad \Psi_-^{\mathrm{IR}} \to \tilde{\Psi}_-^{\mathrm{IR}} \pm e^{2\pi i\mu}\tilde{\Psi}_+^{\mathrm{IR}} \tag{37}$$

and

$$\Psi_+^{\mathrm{IR}} \to \tilde{\Psi}_+^{\mathrm{IR}} \pm e^{2\pi i\mu}\tilde{\Psi}_-^{\mathrm{IR}}, \quad \Psi_-^{\mathrm{IR}} \to \tilde{\Psi}_-^{\mathrm{IR}} \tag{38}$$

hold, respectively. Here the sign $\pm$ *depends on the orientation of the steepest descent paths* $\Gamma_\pm$*.*

6. Summary

As a first step toward the extension of the exact WKB analysis to difference equations, we discuss Bessel's difference equation (B) in this paper. Applying the steepest descent method to integral representations of solutions, we investigate WKB solutions, Stokes curves, and connection formulas for (B). In particular, we show that Stokes curves do really cross and infinitely many new Stokes curves appear from the crossing point for Bessel's difference equation (B).

It is a future problem to extend these results to more general difference equations. One possible next interesting target will be difference equations that Gauss' hypergeometric functions satisfy.

This research is supported by JSPS KAKENHI Grant No. 19H01794.

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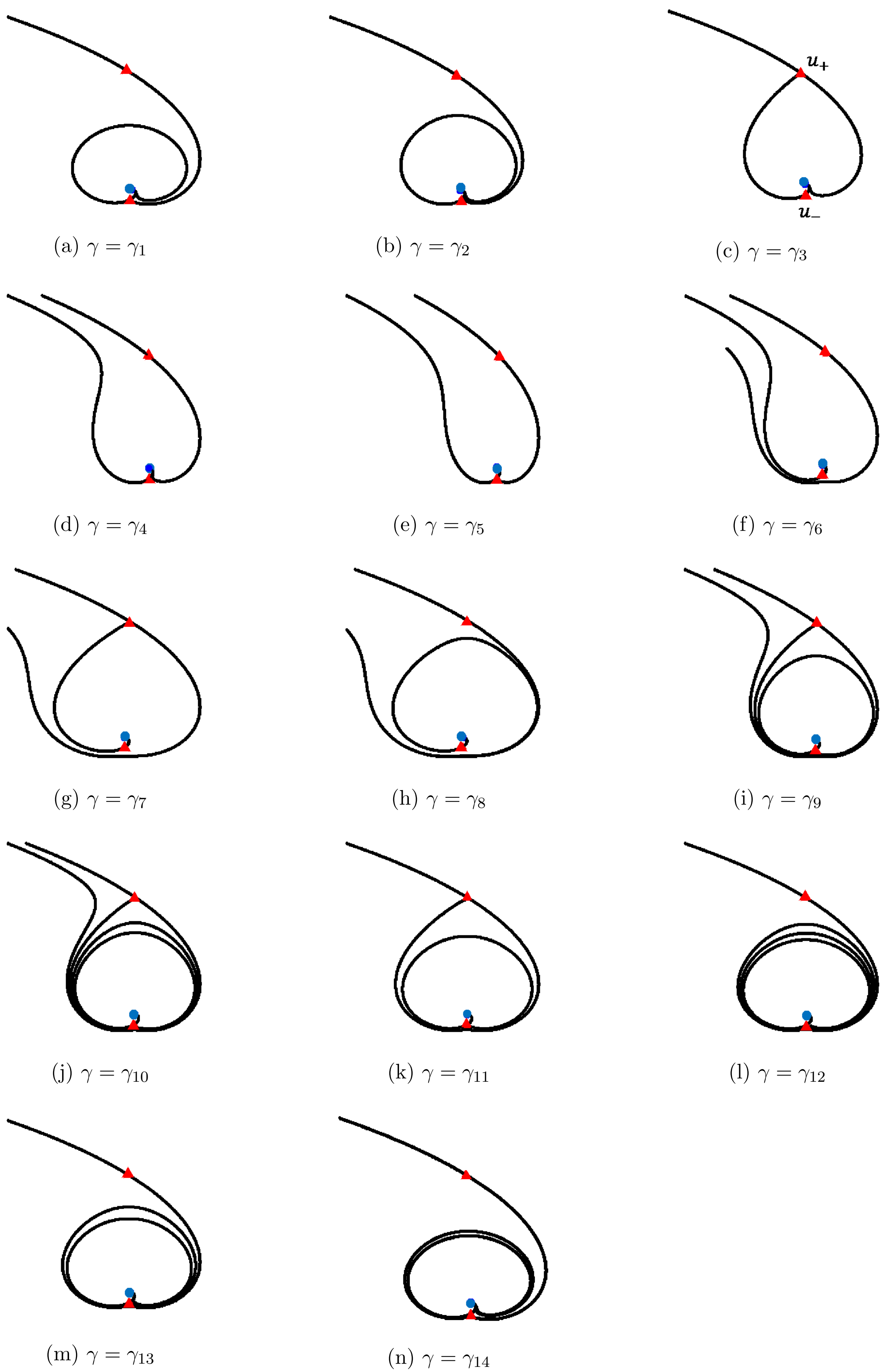

Fig. 5. Steepest descent paths at $\gamma = \gamma_j$ $(1 \leq j \leq 14)$