Studies on linear systems and the eigenvalue problem over the max-plus algebra

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1 Introduction

The max-plus algebra $\mathbb{R} \cup \{-\infty\}$ is a semiring with addition $a \oplus b := \max\{a, b\}$ and multiplication $a \otimes b := a + b$. The study of the max-plus algebra was originated in 1960s with applications to steelworks [20, 21]. After that, the max-plus algebra and similar algebraic structures have been developed under many variant names, such as min-plus algebra, dioid [39], path algebra [38], extremal algebra [79] and tropical semiring [66]. The word *tropical* was introduced in honor of the work by a Brazilian mathematician Imre Simon [70]; Europeans associated Brazil with *tropical*. Now, rich amount of textbooks [9, 13, 22, 40, 42, 43, 57] and surveys [1, 12, 36, 68] on the max-plus algebra are published.

The max-plus algebra has provided powerful tools to the analysis of discrete event systems, which are sometimes represented by Petri nets [61] or heap models [35] and are applied to, e.g., manufacturing [19], rail systems [29] and network calculus [55]. The max-plus algebraic approach to discrete event systems succeeded in evaluating the worst, optimal or average performance [34], computing the invariant space [52], solving the model predictive control problem [28], and so on. A bibliography for applications of the max-plus algebra is found in [53].

For the last two decades, the max-plus algebra has been developed with the connection to the algebraic geometry over fields with valuations. The so called tropical algebraic geometry was motivated in [68, 71, 72] by the theory of Gröbner complexes of ideals and in [31, 59] by the characterization of non-archimedean amoebas. In the tropical geometry, a hypersurface is not the zero set of a max-plus polynomial but the set of the points where the maximum of the polynomial is attained with at least two different terms. This type of *solution* plays a crucial role if we try to obtain max-plus analogues of theories on the conventional algebra.

Another aspect of the max-plus algebra appears in the language of ultradiscrete integrable systems. Integrable systems are special types of differential or difference equations that can be solved exactly and posse a lot of characteristic features: admitting a wide class of symmetries and conservation laws. Ultradiscretization is a kind of limiting process for dependent variables of integrable systems, resulting the systems described by max-plus operations [75]. For example, the box-ball system [74], which is a kind of cellular automata, can be obtained by such process from the Lotka-Volterra equation [75] or the Toda lattice equation [58]. An application of the max-plus eigenvalue problem to these equations is found in [69]. Recently, characterizations of eigenvalues of max-plus tridiagonal matrices via conserved quantities of ultradiscrete integrable systems are derived in [33, 78].

Attempts at overcoming inability of subtraction operation in the max-plus algebra have been made by some researchers. The most familiar one is the symmetrized max-plus algebra [67]. The procedure of symmetrization is analogous to the classical axiomatic construction of negative numbers, that is, every number in the symmetrized max-plus algebra is represented by a pair of two max-plus numbers. Equalities are replaced with the so called balance relations. There is another kind of extension of the max-plus algebra called the supertropical algebra [44, 45, 46, 47, 48, 49, 65]. In this algebra, the concept of the tropical geometry is adopted and special numbers, called ghosts, are introduced so that the sum of two identical numbers behaves as if it were zero. These kinds of extensions were unified into the theory of semirings with symmetry [2, 4]. Some kind of problems concerning max-plus matrix computations are solved in the above frameworks. However, many of the problems related to linear spaces over these algebras, such as independence of vectors or spanning sets of linear spaces, seem to be difficult and remain unsolved.

In the present thesis, the author develops the theory of the max-plus linear algebra, especially the problem for solving linear systems and the eigenvalue problem. As it is mentioned above, solutions of linear systems over the max-plus algebra are sometimes considered in the sense of the tropical geometry. The author gives a combinatorial characterization of solutions of linear systems in terms of the max-plus Cramer's rule [63]. For the eigenvalue problem, analogies to that over the conventional linear algebra have partially been exploited since max-plus square matrices have a very few eigenvalues and/or eigenvectors in general. The author solves this problem by introducing two new concepts that generalize the definition of eigenvectors in some sense. The first ones are *generalized eigenvectors* with respect to eigenvalues, which lead to Jordan canonical forms of max-plus matrices as in the conventional linear algebra [62]. The other concept, *algebraic eigenvectors*, is defined with respect to roots of the max-plus characteristic polynomials of matrices that do not have corresponding eigenvectors [64]. This enables one to deal with those roots, which are as many as the sizes of matrices, as substitutes for eigenvalues.

Section 2 and Section 3 are devoted to give basic definitions and facts on the max-plus linear algebra. A graph theoretic interpretation of a max-plus square matrix is known to be useful for the matrix analysis, especially for the eigenvalue problem. A max-plus square matrix associates the weighted directed graph where the vertices are the row or column indices of the matrix and the edges and their weights correspond to the finite entries. Then, the distance matrix of the corresponding graph coincides with the matrix power series called the Kleene star. In Section 3, it will be observed that a quite different aspect of the max-plus linear space theory to the conventional one. Linear spaces are defined to be sets of vectors that are closed with respect to addition and scalar multiplication in the max-plus algebraic sense. Here, it is notable that *positive* scalars and *negative* ones cannot be distinguished in the max-plus algebra because addition is not invertible and all real numbers are greater than the zero element, $-\infty$. Hence, linear spaces and convex cones over the max-plus algebra become identical. This yields a fundamental result in [16] that the basis, which is the minimal spanning set, if any, of every linear space is unique up to scalar multiples. Further, max-plus analogies of the conventional convex cone theory are studied in [5, 7, 37].

In Section 4, linear systems over the max-plus algebra are discussed. Studies on max-plus linear systems date back to the origin of the max-plus algebra [20], where one-sided systems $A \otimes \boldsymbol{x} = \boldsymbol{b}$ were considered. Another special system $\boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b}$ appeared in 1970s with the relationship to the shortest path problem [38]. Both systems can be solved exactly and numerically in polynomial-time. However, solving general max-plus linear systems is still a hard problem because above two cover quite limited types of systems. For example, since subtraction is not defined in the max-plus algebra, a two-sided linear system $A \otimes \boldsymbol{x} \oplus \boldsymbol{b} = C \otimes \boldsymbol{x} \oplus \boldsymbol{d}$ is not equivalent to a one-sided system. The first algorithm for two-sided systems was given in [15], which was based on the successive elimination of inequalities. A pseudopolynomial-time algorithm, called the alternating method, was proposed in [25]. Despite many efforts [8, 10, 17, 50, 56], polynomial-time algorithms for general two-sided systems have not been invented. On the other hand, a homogeneous linear system $A \otimes \boldsymbol{x} = {}^t(-\infty, \ldots, -\infty)$ only has the trivial solution $\boldsymbol{x} = {}^t(-\infty, \ldots, -\infty)$ in general. Thus, homogeneous systems in the tropical geometric sense are considered instead, i.e., a solution is a vector $\boldsymbol{x}$ such that the maximum of each row of $A \otimes \boldsymbol{x}$ is attained at least twice. Homogeneous systems in the tropical geometric sense can be reduced to two-sided systems. However, the reduction process is complicated and the number of equations becomes much greater. Hence, these two kinds of systems are often investigated separately. The Carmer's rule for homogeneous systems was derived in [68], which is applicable to matrices having one more columns than rows. Recently, it is extended for matrices of arbitrary sizes in [27]. Pseudopolynomial-time algorithms for matrices of arbitrary sizes have been developed in [3, 26, 41]. These methods compute one of the solution of the system, but not all.

A characterization of all solutions of homogeneous systems is one main issue of the present thesis. The set of solutions of a homogeneous system defined by a matrix is called the kernel of the matrix. Since the kernel of every matrix is a max-plus algebraic subspace, it has the unique basis. In the paper [63], the author showed that each vector in the basis of the kernel of a matrix can be computed by applying the tropical analogue of the Cramer's rule to a suitable submatrix of the original matrix. The proof is given by presenting an algorithm for choosing such submatrix, where an analogue of a linkage tree [73] plays an important role. This result is presented in Theorem 4.10 of the thesis. In comparison to the inductive algorithm for computing the basis, e.g., the double description method [5], the contribution of this characterization is to obtain an explicit formula of the vectors in the basis. This analytical expression of the basis will be helpful to reveal properties of solutions of homogeneous linear systems. Similar characterization is found in [5], but it is restricted to matrices whose all square submatrices are nonsingular in the max-algebraic sense, which is often violated. The author succeeded in a characterization of the basis of the kernel of every matrix; the size is arbitrary and the singularities of submatrices are allowable.

Sections 5 is an introduction to the eigenvalue problem over the max-plus algebra. The most fundamental fact is that the maximum eigenvalue of every square matrix is identical to the maximum value of average weights of all circuits in the associated graph. This was already shown in 1960s [21]. The maximum eigenvalue is computed in polynomial-time using the Karp's algorithm [51]. The power algorithm [11], which is also a polynomial-time algorithm, computes the maximum eigenvalue together with a corresponding eigenvector. In the max-plus algebra, it is notable that every irreducible matrix has exactly one eigenvalue, although reducible matrices may have two or more. Basic facts on all eigenvalues and eigenvectors of reducible matrices are summarized in [14]. On the other hand, as in the conventional linear algebra, the characteristic polynomial of a matrix plays an important role in the eigenvalue problem. Every eigenvalue of a matrix is a root of the characteristic polynomial of the matrix, where roots of polynomials are defined in the sense of tropical geometry. In particular, the maximum root coincides with the maximum eigenvalue [23]. However, not all roots of the characteristic polynomial are eigenvalues of the matrix. The meaning of these roots will be given in Section 7.

Section 6 consists of the author's results on the block diagonalization of max-plus matrices. Attempt at diagonalization is investigated in the context of supertropical algebra [47], but there are few research in the max-algebra itself. In general, a max-plus matrix has a few, sometimes just one, eigenvalues and eigenvectors. Hence, it seems difficult to diagonalize max-plus matrices using their eigenvalues and eigenvectors. In the conventional linear algebra, each matrix A can be transformed into a block diagonal matrix, called a Jordan canonical form, even if A is not diagonalizable. If $J = P^{-1}AP$ is a Jordan canonical form of A, the transformation matrix P consists of the basis of the generalized eigenspace of A.

In the paper [62], the author has applied this approach to max-plus matrices. In the max-plus algebra, only generalized permutation matrices have their inverse matrices. Hence, the equality $A \otimes P = P \otimes J$ is considered instead, where P is a nonsingular matrix and J is a block diagonal matrix. The proposed block diagonal matrix is an analogue of a Jordan canonical form in the sense that the corresponding transformation matrix P consists of max-plus generalized eigenvectors. The definition of Jordan canonical forms in Section 6.2 raises two questions: whether a given matrix has a Jordan canonical form, and if it does, how a transformation matrix can be computed. As an answer to the first one, it is shown that a matrix has a Jordan canonical form if and only if each vertex is contained in exactly one spectral circuit in the associated graph. Here, spectral circuits mean the circuits contributing to eigenvalues of the matrix. The proof of this fact also leads to a computational method for a transformation matrix. Further, the obtained transformation matrix has the column space that is identical to the sum of the generalized eigenspaces. This is not an obvious fact in the max-plus algebra because it may be happened that a max-plus subspace is strictly contained into another one with the same dimension. Thus, the result in [62] can be said to specify the best choice of a transformation matrix.

A relationship between eigenvalues of matrices and roots of characteristic polynomials of them is described in Section 7. As in the conventional linear algebra, the characteristic polynomial of an $n \times n$ max-plus matrix admits exactly n root (counting multiplicities). However, the number of eigenvalues of a matrix is at most the number of strongly connected components of the associated graph, which is much smaller than the size of the matrix in general. Hence, many roots of the characteristic polynomial are not eigenvalues of the matrix. Then, the significance of the roots of characteristic polynomials have been clarified by the author in [64]. It is observed that coefficients of the characteristic polynomial of a matrix come from the weights of multi-circuits in the associated graph, where a multi-circuit is the union of disjoint elementary circuits in the graph. So multi-circuits play crucial roles in the study of roots of characteristic polynomials.

Generalizing the equation $A \otimes \boldsymbol{u} = \lambda \otimes \boldsymbol{u}$ for eigenvectors with respect to an eigenvalue λ, the author has introduced the equation

$$(A_{\setminus \mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{u} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus \mathcal{C}}) \otimes \boldsymbol{u} \tag{1.1}$$

for a scalar λ and a multi-circuit $\mathcal{C}$, where matrices $A_{\mathcal{C}}, A_{\setminus \mathcal{C}}, E_{\mathcal{C}}$, and $E_{\setminus \mathcal{C}}$ are determined by $\mathcal{C}$ and defined in Section 7.2. A vector $\boldsymbol{u}$ satisfying (1.1) is called an *algebraic eigenvector* of A with respect to λ. The adjective *algebraic* is taken from Akian et al. [1], in which roots of characteristic polynomials are called *algebraic eigenvalues*. To confirm the validity of the definition of algebraic eigenvectors, it is proved that there exists an algebraic eigenvector with respect to λ if and only if λ is a root of the characteristic polynomial. This holds under the condition that every essential term of the characteristic polynomial is attained with exactly one permutation. This condition is not so strong that it is satisfied by generic matrices, and is also considered in the settings of the supertropical algebra [48]. Further, it is proved that the definition of algebraic eigenvectors does not depend on the choice of a multi-circuit $\mathcal{C}$ whenever it represents the coefficient of the term attaining the maximum in the characteristic polynomial. This leads to the fact that the set of all algebraic eigenvectors with respect to λ is a max-plus subspace, which is called the *algebraic eigenspace*.

Compared to similar approach in the supertropical eigenvalue problem [47, 48], the contribution of the work [64] is an analysis on dimensions of algebraic eigenspaces. Indeed, the definition of supertropical eigenvector, which exploits the tropical geometry, would produce more "eigenvectors" than expected, that is, the number of independent eigenvectors could exceed the multiplicity of the corresponding root. On the other hand, under the condition above and the definition (1.1) of algebraic eigenvectors, the dimension of the algebraic eigenspace is at most the multiplicity of the corresponding root of the characteristic polynomial. This means that algebraic eigenvectors introduced in [64] inherit an important result in the conventional linear algebra.

2 The max-plus algebra

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{\varepsilon\}$ be the set of the real numbers $\mathbb{R}$ together with an extra element $\varepsilon := -\infty$. We define addition $\oplus$ and multiplication $\otimes$ on $\mathbb{R}_{\max}$ in terms of conventional operations by

$$a \oplus b = \max\{a, b\}, \quad a \otimes b = a + b, \quad a, b \in \mathbb{R}_{\max}.$$

These operations satisfy the following properties.

(1) Both $\oplus$ and $\otimes$ are associative: for $a, b, c \in \mathbb{R}_{\max}$,

$$(a \oplus b) \oplus c = a \oplus (b \oplus c), \quad (a \otimes b) \otimes c = a \otimes (b \otimes c).$$

(2) Both $\oplus$ and $\otimes$ are commutative: for $a, b \in \mathbb{R}_{\max}$,

$$a \oplus b = b \oplus a, \quad a \otimes b = b \otimes a.$$

(3) ε is the neutral element for addition: for $a \in \mathbb{R}_{\max}$,

$$a \oplus \varepsilon = \varepsilon \oplus a = a.$$

(4) 0 is the unit element for multiplication: for $a \in \mathbb{R}_{\max}$,

$$a \otimes 0 = 0 \otimes a = a.$$

(5) $-a$ is the inverse of $a \in \mathbb{R}$ for multiplication:

$$a \otimes (-a) = (-a) \otimes a = 0.$$

(6) ε is the absorbing element: for $a \in \mathbb{R}_{\max}$,

$$a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon.$$

(7) Multiplication $\otimes$ is distributive over addition $\oplus$: for $a, b, c \in \mathbb{R}_{\max}$,

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$$

Then, $(\mathbb{R}_{\max}, \oplus, \otimes)$ is a commutative semiring called the *max-plus algebra* or the *tropical semiring*. For details about the max-plus algebra, or the *min-plus algebra*, we refer to textbooks [9, 13, 22, 40, 42, 57].

Let $\mathbb{R}^{m\times n}_{\max}$ be the set of $m \times n$ matrices whose entries are in $\mathbb{R}_{\max}$. The set $\mathbb{R}^{n\times 1}_{\max}$ of column vectors is abbreviated as $\mathbb{R}^{n}_{\max}$. We sometimes denote the (i, j) entry of a matrix A and the ith entry of a vector $\boldsymbol{u}$ by $[A]_{ij}$ and $[\boldsymbol{u}]_i$, respectively. The arithmetic operations on vectors and matrices are defined as those in the conventional linear algebra. For max-plus matrices $A, B \in \mathbb{R}_{\max}^{m\times n}$, we define the matrix sum $A \oplus B \in \mathbb{R}_{\max}^{m\times n}$ by

$$[A \oplus B]_{ij} = [A]_{ij} \oplus [B]_{ij}.$$

For max-plus matrices $A \in \mathbb{R}_{\max}^{l\times m}$ and $B \in \mathbb{R}_{\max}^{m\times n}$, we define the matrix product $A \otimes B \in \mathbb{R}_{\max}^{l\times n}$ by

$$[A \otimes B]_{ij} = \bigoplus_{k=1}^{m} [A]_{ik} \otimes [B]_{kj}.$$

For a max-plus matrix $A \in \mathbb{R}_{\max}^{m\times n}$ and a scalar $\alpha \in \mathbb{R}_{\max}$, we define the scalar multiplication $\alpha \otimes A \in \mathbb{R}_{\max}^{m\times n}$ by

$$[\alpha \otimes A]_{ij} = \alpha \otimes [A]_{ij}.$$

The matrix

$$\mathcal{E} = \begin{pmatrix} \varepsilon & \cdots & \varepsilon \\ \vdots & \ddots & \vdots \\ \varepsilon & \cdots & \varepsilon \end{pmatrix} \in \mathbb{R}_{\max}^{m\times n}$$

is the *zero matrix*, or the *zero vector* if $n = 1$, and the matrix

$$E_n = \begin{pmatrix} 0 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \varepsilon \\ \varepsilon & \cdots & \varepsilon & 0 \end{pmatrix} \in \mathbb{R}_{\max}^{n\times n}$$

is the *identity matrix*.

We define the *determinant* of $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ by

$$\det A = \bigoplus_{\pi\in S_n} \bigotimes_{i=1}^{n} a_{i\pi(i)}, \tag{2.1}$$

where S_n denotes the symmetric group of order n. A matrix $A \in \mathbb{R}_{\max}^{n\times n}$ is called *singular* if the maximum in $\det A$ is attained with at least two permutations; otherwise it is called *nonsingular*.

Example 2.1. Consider a matrix

$$A = \begin{pmatrix} 9 & 8 & 1 \\ 6 & 5 & 3 \\ 0 & 2 & 7 \end{pmatrix}.$$

Since

$$\begin{aligned} &9 \otimes 5 \otimes 7 = 21, \quad 8 \otimes 3 \otimes 0 = 11, \quad 1 \otimes 6 \otimes 2 = 9, \\ &8 \otimes 6 \otimes 7 = 21, \quad 1 \otimes 5 \otimes 0 = 6, \quad\ \ 9 \otimes 3 \otimes 2 = 14, \end{aligned}$$

we see that $\det A = 21$. This value is attained with two permutations: the identity permutation and transposition $(1\ 2)$. Hence, A is singular.

The *inverse* of $A \in \mathbb{R}_{\max}^{n\times n}$ is the matrix $B \in \mathbb{R}_{\max}^{n\times n}$ satisfying $A \otimes B = B \otimes A = E_n$. In the max-plus algebra, only square matrices in a very restricted class have their inverses. A *permutation matrix* associated with a permutation $\pi \in S_n$ is a matrix whose $(i, \pi(i))$ entry is 0 for $i = 1, 2, \dots, n$ and all other entries are ε. A *generalized permutation matrix* is the product of a permutation matrix and a diagonal matrix with finite diagonal entries.

Proposition 2.2. *A matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ has the inverse matrix A^{-1} if and only if it is a generalized permutation matrix.*

Proof. If A is a generalized permutation matrix, i.e., a_{ij} is finite if and only if $j = \pi(i), \pi \in S_n$, then we can easily check that

$$A^{-1} = (a'_{ij}), \quad a'_{ij} = \begin{cases} -a_{ji} & \text{if } i = \pi(j), \\ \varepsilon & \text{otherwise.} \end{cases}$$

Conversely, suppose $B = (b_{ij})$ satisfies $A \otimes B = B \otimes A = E_n$. Then, both A and B have finite entries in every row and in every column. Suppose $b_{jk} \neq \varepsilon$. Then, $a_{ij} = \varepsilon$ for $i \neq k$ by comparing the (i, k) entries of $A \otimes B$ and E_n. Similarly, $a_{k\ell} = \varepsilon$ for $\ell \neq j$ by comparing the (j, ℓ) entries of $B \otimes A$ and E_n. Thus, A contains exactly one finite entry in each row and in each column. This implies that A is a generalized permutation matrix. □

For a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$, we define a weighted digraph $G(A) = (V, E, w)$ as follows. The vertex set and the edge set are $V = \{1, 2, \dots, n\}$ and $E = \{(i, j) \mid a_{ij} \neq \varepsilon\}$, respectively, and the weight function $w : E \to \mathbb{R}$ is defined by $w((i, j)) = a_{ij}$ for $(i, j) \in E$. A sequence $P = (i_0, i_2, \dots, i_\ell)$ of vertices is called an *i_0-i_ℓ path* if $(i_s, i_{s+1}) \in E$ for $s = 0, 1, \dots, \ell - 1$. The number $\ell(P) := \ell$ is called the *length* of P and $w(P) := w((i_0, i_1)) + w((i_1, i_2)) + \cdots + w((i_{\ell-1}, i_\ell))$ is called the *weight* of P. The vertices i_0 and i_ℓ are called the *initial* and the *terminal vertices* of P, respectively. A path is called a *circuit* if its initial and terminal vertices are identical. A circuit $(i_0, i_1, \dots, i_{\ell-1}, i_\ell = i_0)$ is called *elementary* if $i_r \neq i_s$ for $0 \leq r < s \leq \ell - 1$. The *average weight* of a circuit C is defined by $\mathrm{ave}(C) = w(C)/\ell(C)$.

Proposition 2.3. *Let $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$. The (i, j) entry of $A^{\otimes m}$ is identical to the maximum weight of all i-j paths with lengths m. If there is no such path, the (i, j) entry of $A^{\otimes m}$ is ε.*

Proof. We prove by the induction on m. The case $m = 1$ is trivial. Suppose that the assertion is true for $m - 1$. Let P be an i-j path with length m and k be the next vertex of i on P. Since the k-j subpath P' of P has length $m - 1$, we have

$$w(P) = w((i, k)) + w(P') \leq a_{ik} + [A^{\otimes m-1}]_{kj}.$$

Hence, taking the maximum weight i-j path $\tilde{P}$ with length m, we get

$$w(\tilde{P}) = \max_P w(P) \le \bigoplus_{k=1}^{n} a_{ik} \otimes [A^{\otimes m-1}]_{kj},$$

where the maximum in the middle is taken over all i-j paths with lengths m. Since there exists an i-j path with length m attaining the most right-hand side as its weight, we have

$$w(\tilde{P}) = \bigoplus_{k=1}^{n} a_{ik} \otimes [A^{\otimes m-1}]_{kj} = [A^{\otimes m}]_{ij},$$

proving the assertion for m. $\square$

For $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$, we consider the matrix power series of the form

$$A^* := E_n \oplus A \oplus A^{\otimes 2} \oplus \cdots ,$$

which is called the *Kleene star* of A. If $G(A)$ has no circuit with positive weight, this infinite sum terminates as

$$A^* = E_n \oplus A \oplus A^{\otimes 2} \oplus \cdots \oplus A^{\otimes n-1}$$

since i-j paths attaining the maximum weight can be assumed to have at most $n-1$ edges for any i, j. In this case, (i, j) entry of A^* is the maximum weight of all i-j paths with any lengths. Similarly, the matrix power series $A^+ := A \oplus A^{\otimes 2} \oplus \cdots$, called the *Kleene plus* of A, terminates as $A^+ = A \oplus A^{\otimes 2} \oplus \cdots A^{\otimes n}$ if $G(A)$ has no circuit with positive weight. The computational complexities of A^* and A^+ are $O(n^3)$ due to the Floyd-Warshall algorithm [32, 77].

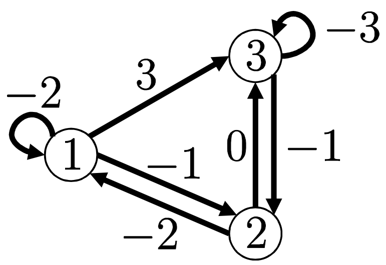

Figure 1: Graph associated with the matrix A in Example 2.4

Example 2.4. Consider a matrix

$$A = \begin{pmatrix} -2 & -1 & 3 \\ -2 & \varepsilon & 0 \\ \varepsilon & -1 & -3 \end{pmatrix}.$$

Figure 1 shows the associated graph $G(A)$. Since $G(A)$ has no circuit with positive weight, we compute $A^{\otimes 2}$ and A^* as

$$A^{\otimes 2} = A \otimes A = \begin{pmatrix} -3 & 2 & 1 \\ -4 & -1 & 1 \\ -3 & -4 & -1 \end{pmatrix},$$

$$A^* = E_3 \oplus A \oplus A^{\otimes 2} = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 1 \\ -3 & -1 & 0 \end{pmatrix}.$$

3 Linear spaces over the max-plus algebra

In this section, we summarize the theory of max-plus linear spaces mainly presented in the book [13].

A subset $U \subset \mathbb{R}_{\max}^n$ is called a *subspace* if it satisfies

(1) $\boldsymbol{u} \oplus \boldsymbol{v} \in U$ for $\boldsymbol{u}, \boldsymbol{v} \in U$, and

(2) $c \otimes \boldsymbol{u} \in U$ for $\boldsymbol{u} \in U$ and $c \in \mathbb{R}_{\max}$.

For a subspace $U \subset \mathbb{R}_{\max}^n$, a subset $F \subset U$ *spans* U if for all $\boldsymbol{u} \in U$ there exists a finite subset F' of F and scalars $c_{\boldsymbol{v}}, \boldsymbol{v} \in F'$, such that

$$\boldsymbol{u} = \bigoplus_{\boldsymbol{v} \in F'} c_{\boldsymbol{v}} \otimes \boldsymbol{v}.$$

The subspace spanned by $F \subset \mathbb{R}_{\max}^n$ is denoted by $\mathrm{span}(F)$. A subset F is called *dependent* if there exist a vector $\boldsymbol{u} \in F$ such that $\boldsymbol{u} \in \mathrm{span}(F \setminus \{\boldsymbol{u}\})$; otherwise it is called *independent*. The independent spanning subset of a subspace is called a *basis* of it.

Let $F \subset \mathbb{R}_{\max}^n$. A vector $\boldsymbol{u} \in F$ is called an *extreme vector* of F if $\boldsymbol{u} = \boldsymbol{v} \oplus \boldsymbol{w}$ for $\boldsymbol{v}, \boldsymbol{w} \in F$ implies $\boldsymbol{u} = \boldsymbol{v}$ or $\boldsymbol{u} = \boldsymbol{w}$. A vector is called *scaled* if its maximum entry is 0. A subset is also called *scaled* if all vectors in it are scaled.

Lemma 3.1. *Let $U \subset \mathbb{R}_{\max}^n$ be a subspace and F be a scaled spanning set of U. If $\boldsymbol{u} \in U$ is a scaled extreme vector of U, then $\boldsymbol{u} \in F$.*

Proof. Since F is a spanning set of U, there exists a finite subset F' of F and scalars $c_{\boldsymbol{v}}, \boldsymbol{v} \in F'$ such that

$$\boldsymbol{u} = \bigoplus_{\boldsymbol{v} \in F'} c_{\boldsymbol{v}} \otimes \boldsymbol{v}.$$

As $\boldsymbol{u}$ is an extreme vector, $\boldsymbol{u} = c_{\boldsymbol{u}_i} \otimes \boldsymbol{u}_i$ for some i. Thus, $\boldsymbol{u} = \boldsymbol{u}_i \in F$ since both $\boldsymbol{u}$ and $\boldsymbol{u}_i$ are scaled. $\square$

Lemma 3.2. *The set of all scaled extreme vectors of a subspace $U \subset \mathbb{R}_{\max}^n$ is independent.*

Proof. Let F be the set of all scaled extreme vectors of U. On the contrary, suppose that $F \neq \emptyset$ is dependent. Then, there is a vector $\boldsymbol{u} \in F$ such that $\boldsymbol{u} \in \mathrm{span}(F \setminus \{\boldsymbol{u}\})$. Since $\mathrm{span}(F \setminus \{\boldsymbol{u}\}) \subset U$, $\boldsymbol{u}$ is also an extreme vector of $\mathrm{span}(F \setminus \{\boldsymbol{u}\})$. From Lemma 3.1, $\boldsymbol{u} \in F \setminus \{\boldsymbol{u}\}$, which is a contradiction. Thus, F is an independent set. $\square$

We define the support of a vector $\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n) \in \mathbb{R}_{\max}^n$ by

$$\mathrm{supp}(\boldsymbol{u}) = \{i \mid u_i \neq \varepsilon\}.$$

Lemma 3.3. *For any subset $F \subset \mathbb{R}^n_{\max}$, (1) and (2) are equivalent.*

(1) $\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n) \in span(F)$.

(2) For all $j \in supp(\boldsymbol{u})$, there exist vectors $\boldsymbol{v}^j \in F$ such that $(u_j - [\boldsymbol{v}^j]_j) \otimes \boldsymbol{v}^j \leq \boldsymbol{u}$.

Proof. (1) $\Rightarrow$ (2): Suppose $\boldsymbol{u} \in \mathrm{span}(F)$. Then, there is a finite subset F' of F and scalars $c_{\boldsymbol{v}}, \boldsymbol{v} \in F'$ such that

$$\boldsymbol{u} = \bigoplus_{\boldsymbol{v} \in F'} c_{\boldsymbol{v}} \otimes \boldsymbol{v}.$$

We have $\boldsymbol{u} \geq c_{\boldsymbol{v}} \otimes \boldsymbol{v}$ for $\boldsymbol{v} \in F'$. For any $j \in \mathrm{supp}(\boldsymbol{u})$, there is at least one $\boldsymbol{v} \in F'$ such that $u_j = c_{\boldsymbol{v}} \otimes [\boldsymbol{v}]_j$. We set $\boldsymbol{v}^j = \boldsymbol{v}$. Then, we have $(u_j - [\boldsymbol{v}^j]_j) \otimes \boldsymbol{v}^j \leq \boldsymbol{u}$.

(2) $\Rightarrow$ (1): Taking such $\boldsymbol{v}^j \in F$ for $j \in \mathrm{supp}(\boldsymbol{u})$, we have

$$\boldsymbol{u} = \bigoplus_{j \in \mathrm{supp}(\boldsymbol{u})} (u_j - [\boldsymbol{v}^j]_j) \otimes \boldsymbol{v}^j \in \mathrm{span}(F).$$

□

Lemma 3.4. *Let $U \subset \mathbb{R}^n_{\max}$ be a subspace and F be a subset of U. If $\boldsymbol{u}$ is not an extreme vector of $\boldsymbol{u}$, then $F \setminus \{\boldsymbol{u}\}$ spans U.*

Proof. Suppose $\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n) \in F$ is not an extreme vector of $\boldsymbol{u}$. For all $j \in \mathrm{supp}(\boldsymbol{u})$, there exists a vector $\boldsymbol{v}^j \in U$ such that $(u_j - [\boldsymbol{v}^j]_j) \otimes \boldsymbol{v}^j \leq \boldsymbol{u}$ and $(u_j - [\boldsymbol{v}^j]_j) \otimes \boldsymbol{v}^j \neq \boldsymbol{u}$. Indeed, if $(u_j - [\boldsymbol{v}^j]_j) \otimes \boldsymbol{v} \leq \boldsymbol{u}$ implied $(u_j - [\boldsymbol{v}^j]_j) \otimes \boldsymbol{v} = \boldsymbol{u}$ for some $j \in \mathrm{supp}(\boldsymbol{u})$, $\boldsymbol{u}$ would be an extreme vector of U, leading to a contradiction. Since $(u_j - [\boldsymbol{v}^j]_j) \otimes \boldsymbol{v}^j \in \mathrm{span}(F)$, there is a vector $\boldsymbol{w}^j \in F$ such that $(u_j - [\boldsymbol{w}^j]_j) \otimes \boldsymbol{w}^j \leq (u_j - [\boldsymbol{v}^j]_j) \otimes \boldsymbol{v}^j \leq \boldsymbol{u}$ by Lemma 3.3. From the proof of the same lemma,

$$\boldsymbol{u} = \bigoplus_{j \in \mathrm{supp}(\boldsymbol{u})} (u_j - [\boldsymbol{w}^j]_j) \otimes \boldsymbol{w}^j.$$

Since $(u_j - [\boldsymbol{w}^j]_j) \otimes \boldsymbol{w}^j \neq \boldsymbol{u}$ for $j \in \mathrm{supp}(\boldsymbol{u})$ from our construction, we see that $\boldsymbol{u} \in \mathrm{span}(F \setminus \{\boldsymbol{u}\})$. Thus, $F \setminus \{\boldsymbol{u}\}$ spans U. □

Theorem 3.5 ([16])**.** *Let $U \subset \mathbb{R}^n_{\max}$ be a subspace and F be the set of all scaled extreme vectors of U. For a scaled subset F' of U, the followings are equivalent.*

(1) F' is a minimal spanning set of U.

(2) $F' = F$ and F' spans U.

(3) F' is a basis of U.

Proof. (1) $\Rightarrow$ (2): If $F' \neq F$, there would be a vector $\boldsymbol{u} \in F' \setminus F$. From above lemma, $F' \setminus \{\boldsymbol{u}\}$ would also be a spanning set of U, contradicting the minimality of F'.

(2) $\Rightarrow$ (3): This follows from the independence of F, which is proved in Lemma 3.2.

(3) $\Rightarrow$ (1): An independent spanning set must be minimal. □

If a subspace $U \subset \mathbb{R}^n_{\max}$ is spanned by a finite subset of U, it has a minimal spanning set. Hence, U has a basis consisting of all scaled extreme vectors of U. In particular, a basis of U is uniquely determined up to scalar multiples. We call the number of vectors in a basis the *dimension* of U. Once a finite spanning set F of subspace U is given, the extreme vectors of U can be easily detected.

Proposition 3.6. *Let $F = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\} \subset \mathbb{R}^n_{\max}$ be a spanning set of a subspace U, no vector of which is a scalar multiple of another one. Fix any vector $\boldsymbol{u}_k \in F$ and let c_j be the maximum value such that $c_j \otimes \boldsymbol{u}_j \leq \boldsymbol{u}_k$ for $j \neq k$. Then, $\boldsymbol{u}_k$ is an extreme vector of U if and only if*

$$\boldsymbol{u}_k \neq \bigoplus_{j \neq k} c_j \otimes \boldsymbol{u}_j.$$

In the end of this section, we mention ranks of matrices. The *column space* of a matrix is the subspace spanned by the columns of that matrix. The dimension of the column space is called the *column rank* of the matrix. The next proposition can be derived from the inequality for various kinds of ranks of matrices [2, Section 8], but we give a self-contained proof. We also refer to [30] for details about ranks of matrices.

Proposition 3.7. *Let $A \in \mathbb{R}^{n \times n}_{\max}$. If the column rank of A is less than n, then A is singular.*

Proof. We first note that a matrix is singular if it has two identical columns up to scalar multiples. Indeed, if the ith column is a scalar multiple of the jth column, then π and $\pi \circ (ij)$ attain the same values in (2.1) for any $\pi \in S_n$. Suppose the ith column A_i of A is expressed as

$$A_i = c_{i1} \otimes \boldsymbol{u}_1 \oplus c_{i2} \otimes \boldsymbol{u}_2 \oplus \cdots \oplus c_{ir} \otimes \boldsymbol{u}_r, \quad i = 1, 2, \ldots, n,$$

where r is the dimension of the column space of A. Then, the determinant of A is computed as

$$\det A = \bigoplus_{i_1, i_2, \ldots, i_n = 1}^{r} \det(c_{1i_1} \otimes \boldsymbol{u}_{i_1}, c_{2i_2} \otimes \boldsymbol{u}_{i_2}, \ldots, c_{ni_n} \otimes \boldsymbol{u}_{i_n}).$$

Since $r < n$ from the assumption, each determinant in the right-hand side is attained with at least two permutations, and hence so $\det A$ is. Thus, we conclude that A is singular. □

Example 3.8. Consider the matrix

$$A = \begin{pmatrix} -1 & -2 & 4 & 1 \\ -2 & 0 & -1 & -1 \\ 3 & -2 & -5 & 1 \\ 0 & -3 & -2 & -2 \end{pmatrix}.$$

Fixing the fourth column ${}^t(1, -1, 1, 2)$, we have $c_1 = -2, c_2 = -1$ and $c_3 = -3$ in Proposition 3.6. Since

$$(-2) \otimes \begin{pmatrix} -1 \\ -2 \\ 3 \\ 0 \end{pmatrix} \oplus (-1) \otimes \begin{pmatrix} -2 \\ 0 \\ -2 \\ -3 \end{pmatrix} \oplus (-3) \otimes \begin{pmatrix} 4 \\ -1 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix},$$

the columns of A are dependent. Indeed, $\det A = -2$ is attained with several permutations, such as (1234), (14)(23), and so on.

4 Linear systems and their solutions

In this section, we discuss linear systems over the max-plus algebra. We will see in Section 4.1 that two-sided nonhomogeneous systems $\boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b}$ can be solved in terms of Kleene stars of matrices. The method is an analogue of the Jacobi iterative method in the (conventional) numerical linear algebra, but in the case of the max-plus algebra, the iteration terminates in finite steps, giving an exact solution. In Section 4.2, we give a brief introduction to tropical linear systems. In particular, kernels of matrices are defined as the solutions of homogeneous tropical linear systems. For $(n-1) \times n$ matrices, a solution of a tropical linear system can be found by using the max-plus analogue of the Cramer's rule. The Cramer's rule was first proved in [68] using the knowledge of the tropical algebraic geometry. In Secttion 4.3, however, we will present a self-contained proof. Section 4.4 is the first main result of the present thesis, which is shown in [63]. We focus on not one of the solutions but all solutions of tropical linear systems. For each vector of the basis of the kernel of a matrix, we give a characterization in terms of the Cramer's rule. This characterization also suggests the way to compute the basis of the kernel.

4.1 A classical two-sided system: $\boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b}$

For $A \in \mathbb{R}_{\max}^{n \times n}$ and $\boldsymbol{b} \in \mathbb{R}_{\max}^{n}$, we consider the system of the form

$$\boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b}. \tag{4.1}$$

This is easily solved by iteration.

Proposition 4.1 ([38]). *Let $A \in \mathbb{R}_{\max}^{n \times n}$ and $\boldsymbol{b} \in \mathbb{R}_{\max}^{n}$. If the graph $G(A)$ has no circuit with positive weight, then $\boldsymbol{x} = A^* \otimes \boldsymbol{b}$ is a solution of (4.1). Further, if all circuits in $G(A)$ have negative weights, then the solution is unique.*

Proof. If the graph $G(A)$ has no circuit with positive weight, then both A^* and A^+ converge to matrices in $\mathbb{R}_{\max}^{n \times n}$. We compute

$$\begin{aligned} A \otimes (A^* \otimes \boldsymbol{b}) \oplus \boldsymbol{b} &= A^+ \otimes \boldsymbol{b} \oplus \boldsymbol{b} \\ &= A^* \otimes \boldsymbol{b}, \end{aligned}$$

which proves the first statement of the proposition.

Next, assume that all circuits in $G(A)$ have negative weights. If $\boldsymbol{x}^*$ is a solution, then we have

$$\boldsymbol{x}^* = A^{\otimes m} \otimes \boldsymbol{x}^* \oplus A^* \otimes \boldsymbol{b}$$

for all $m \geq n$ by iterative substitution. Taking the limit $m \to \infty$, we see that all entires of $A^{\otimes m}$ go to ε since these entires express the maximum weights of paths with lengths m. Thus, we have $\boldsymbol{x}^* = A^* \otimes \boldsymbol{b}$. ☐

This is easily extended to the following case.

Corollary 4.2. *Let $P \in \mathbb{R}_{\max}^{n\times n}$ be a generalized permutation matrix, $Q \in \mathbb{R}_{\max}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}_{\max}^{n}$. Then the system*

$$P \otimes \boldsymbol{x} = Q \otimes \boldsymbol{x} \oplus \boldsymbol{b}$$

has a solution $\boldsymbol{x} = (P^{-1} \otimes Q)^* \otimes (P^{-1} \otimes \boldsymbol{b})$ if $G(P^{-1} \otimes Q)$ has no circuit with positive weight. If all circuits in $G(P^{-1} \otimes Q)$ has negative weights, then the solution is unique.

4.2 Tropical linear systems

We consider a *linear form* of the variables $x_1, x_2, \dots, x_n$ of the form

$$f = a_1 \otimes x_1 \oplus a_2 \otimes x_2 \oplus \cdots \oplus a_n \otimes x_n. \tag{4.2}$$

The *tropical hyperplane* defined by a linear form f, denoted by $T(f)$, is the set of all vectors $\boldsymbol{x} = {}^t(x_1, x_2, \dots, x_n) \in \mathbb{R}_{\max}^n$ such that the maximum of the terms on the right-hand side of (4.2) is attained at least twice. A vector in $T(f)$ is also called a *solution* of f. You may feel that the word *solution* is strange since f is not an equality. This is understood by the following observation: $\boldsymbol{x}$ is a solution of f if and only if there exist an index k such that $\boldsymbol{x}$ satisfies

$$\bigoplus_{j \neq k} a_j \otimes x_j = a_{ik} \otimes x_k,$$

which means $\boldsymbol{x}$ is a solution if we *transpose* some term in $f = \varepsilon$. The word *solution* also deserves a description in terms of a valuation on a field K. A function $\mathrm{val} : K \to \mathbb{R} \cup \{\infty\}$ satisfying the following three properties is called a *valuation* on K:

(1) $\mathrm{val}(a) = \infty$ if and only if $a = 0$,

(2) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$ for $a, b \in \mathbb{R}$, and

(3) $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for $a, b \in \mathbb{R}$.

If ${}^t(X_1, X_2, \dots, X_n) \in K^n$ is a solution of the usual equation $A_1X_1 + A_2X_2 + \cdots + A_nX_n = 0$, then ${}^t(x_1, x_2, \dots, x_n) \in \mathbb{R}_{\max}^n$ is a solution of a max-plus linear form (4.2) where $a_i = -\mathrm{val}(A_i)$ and $x_i = -\mathrm{val}(X_i)$ for $i = 1, 2, \dots, n$. Otherwise, $-\mathrm{val}(A_1X_1 + A_2X_2 + \cdots + A_nX_n) = a_1 \otimes x_1 \oplus a_2 \otimes x_2 \oplus \cdots \oplus a_n \otimes x_n$.

Example 4.3. Let $f = 4 \otimes x \oplus 0 \otimes y \oplus 5 \otimes z$. The hyperplane defined by f is

$$\begin{aligned} T(f) = &\{{}^t(x, y, z) \in \mathbb{R}_{\max}^3 \mid 4 + x = 5 + z,\ y \leq 5 + z\} \\ &\cup \{{}^t(x, y, z) \in \mathbb{R}_{\max}^3 \mid y = 5 + z,\ 4 + x \leq 5 + z\} \\ &\cup \{{}^t(x, y, z) \in \mathbb{R}_{\max}^3 \mid 4 + x = y,\ 4 + x \geq 5 + z\}. \end{aligned}$$

The projection of $T(f)$ onto the plane $z = 0$ is a union of three rays, see Figure 2.

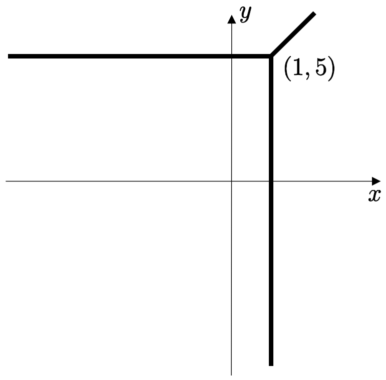

Figure 2: Tropical hyperplane of Example 4.3 projected onto $z = 0$.

Given a set of linear forms $\{f_1, f_2, \ldots, f_m\}$, the intersection of the tropical hyperplanes $T(f_1), T(f_2), \ldots, T(f_m)$ is called a *tropical linear prevariety*. For $A = (a_{ij}) \in \mathbb{R}_{\max}^{m\times n}$, let $T(A)$ denote the intersection of m hyperplanes $T(a_{i1} \otimes x_1 \oplus a_{i2} \otimes x_2 \oplus \cdots \oplus a_{in} \otimes x_n)$, i.e.,

$$T(A) = \bigcap_{i=1}^{m} T(a_{i1} \otimes x_1 \oplus a_{i2} \otimes x_2 \oplus \cdots \oplus a_{in} \otimes x_n).$$

The tropical linear prevariety $T(A)$ is also called the *kernel* of A. From the viewpoint of linear systems, vectors in kernel $T(A)$ are called *solutions* of the *tropical linear system* $A \otimes \boldsymbol{x}$. We can easily verify that the kernel of every matrix is a subspace of $\mathbb{R}_{\max}^n$. As in the conventional linear algebra, the triviality of the kernel of a square matrices is equivalent to the nonsingularity of the matrix.

Proposition 4.4 ([3]). *A square matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ is nonsingular if and only if $T(A) = \{\mathcal{E}\}$.*

For a square matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}_{\max}^n$, let us consider the tropical linear system

$$(A\ \boldsymbol{b}) \otimes \tilde{\boldsymbol{x}}, \tag{4.3}$$

where $\tilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ y \end{pmatrix} = {}^t(x_1, x_2, \ldots, x_n, y)$. If $\det A \neq \varepsilon$, fix a permutation $\pi_A \in S_n$ attaining the maximum of $\det A$. We define two matrices $P_A, Q_A \in \mathbb{R}_{\max}^{n\times n}$ by

$$[P_A]_{ij} = \begin{cases} a_{ij} & \text{if } j = \pi_A(i), \\ \varepsilon & \text{otherwise,} \end{cases} \quad [Q_A]_{ij} = \begin{cases} \varepsilon & \text{if } j = \pi_A(i), \\ a_{ij} & \text{otherwise,} \end{cases}$$

We can easily prove that a vector $\boldsymbol{x} \in \mathbb{R}^n_{\max}$ is a solution of

$$P_A \otimes \boldsymbol{x} = Q_A \otimes \boldsymbol{x} \oplus \boldsymbol{b} \tag{4.4}$$

if and only if $\tilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ 0 \end{pmatrix}$ is a solutions of (4.3). The following lemma shows that the system (4.4) can be solved using the technique in Section 4.1.

Lemma 4.5. *With the above notations, the graph $G(P_A^{-1} \otimes Q_A)$ has no circuit with positive weight.*

Proof. Let $[P_A^{-1} \otimes Q_A]_{ij}$ be the (i,j) entry of $P_A^{-1} \otimes Q_A$. We compute

$$[P_A^{-1} \otimes Q_A]_{ij} = \begin{cases} \varepsilon & \text{if } i = j, \\ a_{\pi_A^{-1}(i)j} - a_{\pi_A^{-1}(i)i} & \text{otherwise.} \end{cases}$$

Take an elementary circuit $(i_0, i_1, \ldots, i_{\ell-1}, i_\ell = i_0)$ in $G(P_A^{-1} \otimes Q_A)$ and let $\sigma \in S_n$ be a cyclic permutation $(i_0 i_1 \cdots i_{\ell-1})$. Since $\sigma(j) = j$ for $j \notin \{i_0, i_1, \ldots, i_{\ell-1}\}$, the weight of this circuit is

$$\begin{aligned}
\sum_{k=0}^{\ell-1} [P_A^{-1} \otimes Q_A]_{i_k i_{k+1}} &= \sum_{k=0}^{\ell-1} (a_{\pi_A^{-1}(i_k) i_{k+1}} - a_{\pi_A^{-1}(i_k) i_k}) \\
&= \sum_{j=1}^{n} (a_{\pi_A^{-1}(j)\sigma(j)} - a_{\pi_A^{-1}(j)j}) \\
&= \sum_{j=1}^{n} (a_{j\sigma(\pi_A(j))} - a_{j\pi_A(j)}) \\
&= \sum_{j=1}^{n} a_{j\sigma(\pi_A(j))} - \det A \\
&\leq 0.
\end{aligned}$$

This holds for any elementary circuit, which proves the lemma. $\square$

Theorem 4.6. *Let $A \in \mathbb{R}^{n \times n}_{\max}$ and $\boldsymbol{b} \in \mathbb{R}^n_{\max}$ and suppose* $\det A \neq \varepsilon$. *Then, the vector*

$$\boldsymbol{x} = \begin{pmatrix} (P_A^{-1} \otimes Q_A)^* \otimes (P_A^{-1} \otimes \boldsymbol{b}) \\ 0 \end{pmatrix}$$

is a solution of the tropical linear system (4.3).

Proof. This follows from Corollary 4.2 and Lemma 4.5. $\square$

4.3 The Cramer's rule

We describe the max-plus analogue of the Cramer's rule. For a matrix $A \in \mathbb{R}_{\max}^{m\times n}$, let $A^{(i)} \in \mathbb{R}_{\max}^{m\times(n-1)}$ denote the matrix obtained from A by removing the ith column. We define the *Cramer vector* of $A \in \mathbb{R}_{\max}^{n\times(n+1)}$ by

$$\boldsymbol{x}^{\mathrm{Cram},A} = {}^t\!\left(\det(A^{(1)}), \det(A^{(2)}), \ldots, \det(A^{(n+1)})\right).$$

The Cramer's rule for tropical linear systems is stated as follows.

Theorem 4.7 ([68])**.** *Let $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times(n+1)}$. Then, the Cramer vector of A is a solution of the tropical linear system $A \otimes \boldsymbol{x}$.*

Proof. Let us consider the ith row of $A \otimes \boldsymbol{x}^{\mathrm{Cram},A}$:

$$a_{i1} \otimes \det(A^{(1)}) \oplus a_{i2} \otimes \det(A^{(2)}) \oplus \cdots \oplus a_{i\,n+1} \otimes \det(A^{(n+1)}). \tag{4.5}$$

If (4.5) is ε for $i = 1, 2, \ldots, n$, the assertion of the theorem immediately follows. For any index i such that (4.5) is finite, assume that the maximum in (4.5) is attained with $a_{ik} \otimes \det(A^{(k)})$ and

$$\det(A^{(k)}) = a_{1\sigma(1)} \otimes a_{2\sigma(2)} \otimes \cdots \otimes a_{n\sigma(n)},$$

where $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n+1\} \setminus \{k\}$ is a bijection. Then, we have

$$\begin{aligned}
a_{ik} \otimes \det(A^{(k)}) &= a_{ik} \otimes \left(a_{1\sigma(1)} \otimes a_{2\sigma(2)} \otimes \cdots \otimes a_{n\sigma(n)}\right) \\
&= a_{i\sigma(i)} \otimes \left(a_{1\sigma(1)} \otimes a_{2\sigma(2)} \otimes \cdots \otimes a_{ik} \otimes \cdots \otimes a_{n\sigma(n)}\right) \\
&\le a_{i\sigma(i)} \otimes \det(A^{(\sigma(i))}) \\
&\le a_{ik} \otimes \det(A^{(k)}).
\end{aligned}$$

Hence, the maximum in (4.5) is also attained with $a_{i\sigma(i)} \otimes \det(A^{(\sigma(i))})$, which means $\boldsymbol{x}^{\mathrm{Cram},A} \in T(A)$. □

Example 4.8. For a matrix $A = \begin{pmatrix} 4 & 0 & 5 \\ 5 & 6 & 2 \end{pmatrix}$, we consider the tropical linear system $A \otimes \boldsymbol{x}$. Then $T(A)$ is the intersection of two tropical hyperplanes defined by $f = 4 \otimes x \oplus 0 \otimes y \oplus 5 \otimes z$ and $g = 5 \otimes x \oplus 6 \otimes y \oplus 2 \otimes z$. It consists of a line $\{c \otimes {}^t(1, 0, 0) \mid c \in \mathbb{R}_{\max}\}$, see Figure 3. This is also obtained from the Cramer's rule:

$$\begin{aligned}
&{}^t(\det(A^{(1)}), \det(A^{(2)}), \det(A^{(3)})) \\
=&{}^t\!\left(\det \begin{pmatrix} 0 & 5 \\ 6 & 2 \end{pmatrix}, \det \begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix}, \det \begin{pmatrix} 4 & 0 \\ 5 & 6 \end{pmatrix}\right) \\
=&{}^t(11, 10, 10) \\
=&10 \otimes {}^t(1, 0, 0).
\end{aligned}$$

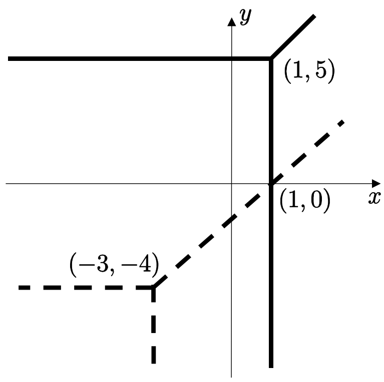

Figure 3: The intersection of two tropical hyperplanes of Example 4.8 projected onto $z = 0$.

4.4 Characterization of kernels of matrices

We first see that the kernel of every matrix is a subspace spanned by a finite set of vectors.

Proposition 4.9. *The kernel $T(A)$ of $A \in \mathbb{R}_{\max}^{m\times n}$ is a subspace of $\mathbb{R}_{\max}^{n}$ that has a finite spanning set.*

Proof. For the proof, we refer to the tropical double description method in [5]. Since $\mathbb{R}_{\max}^{n}$ is spanned by n standard basis vectors, it suffices, by induction, to show that if U is spanned by a finite set, then so $T(f) \cap U$ is for any linear form $f(\boldsymbol{x}) = a_1 \otimes x_1 \oplus a_2 \otimes x_2 \oplus \cdots \oplus a_n \otimes x_n$. Suppose U is spanned by $F = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}$. We will see that $T(f) \cap U$ is spanned by

$$\begin{aligned}
&(T(f) \cap F) \\
&\cup \{f(\boldsymbol{u}_k) \otimes \boldsymbol{u}_j \oplus f(\boldsymbol{u}_j) \otimes \boldsymbol{u}_k \mid \boldsymbol{u}_j \in F \cap T(f), \boldsymbol{u}_k \in F \setminus T(f)\} \\
&\cup \left\{ f(\boldsymbol{u}_k) \otimes \boldsymbol{u}_j \oplus f(\boldsymbol{u}_j) \otimes \boldsymbol{u}_k \;\middle|\; \begin{array}{l} \boldsymbol{u}_j, \boldsymbol{u}_k \in F \setminus T(f), \\ \text{the maximums of } f(\boldsymbol{u}_j) \text{ and } f(\boldsymbol{u}_k) \\ \text{are attained with different indices} \end{array} \right\}.
\end{aligned} \tag{4.6}$$

The last two sets in (4.6) are contained in $T(f)$. Indeed, since we have

$$f(f(\boldsymbol{u}_k) \otimes \boldsymbol{u}_j) = f(f(\boldsymbol{u}_j) \otimes \boldsymbol{u}_k) = f(\boldsymbol{u}_j) \otimes f(\boldsymbol{u}_k),$$

the maximum $f(f(\boldsymbol{u}_k) \otimes \boldsymbol{u}_j \oplus f(\boldsymbol{u}_j) \otimes \boldsymbol{u}_k)$ is attained at least twice. So it remains to show that all vectors in $T(f) \cap U$ are represented by linear combinations of vectors in (4.6).

Take any vector $\boldsymbol{v} \in T(f) \cap U$. Then, it is expressed as

$$\boldsymbol{v} = \bigoplus_{i=1}^{r} c_i \otimes \boldsymbol{u}_i.$$

Let I_1 and I_2 be the sets of indices such that $\boldsymbol{u}_i \in T(f)$ for all $i \in I_1$ and $\boldsymbol{u}_i \notin T(f)$ for all $i \in I_2$. If $f(\boldsymbol{v}) = \varepsilon$, then c_i must be ε for $i \in I_2$ because $f(\boldsymbol{u}_i) \neq \varepsilon$ for those i. Hence, $\boldsymbol{v}$ is a linear combinations of vectors in $T(f) \cap F$. Next, assume that $f(\boldsymbol{v}) \neq \varepsilon$. If there is an index $i_0 \in I_1$ such that $f(\boldsymbol{v}) = c_{i_0} \otimes f(\boldsymbol{u}_{i_0})$, then

$$\bigoplus_{i \in I_1} c_i \otimes \boldsymbol{u}_i = \bigoplus_{i \in I_1} c_i \otimes \boldsymbol{u}_i \oplus (-f(\boldsymbol{v})) \otimes \left(\bigoplus_{i \in I_2} c_i \otimes f(\boldsymbol{u}_i) \right) \otimes c_{i_0} \otimes \boldsymbol{u}_{i_0}$$

and

$$\bigoplus_{i \in I_2} c_i \otimes \boldsymbol{u}_i = (-f(\boldsymbol{v})) \otimes c_{i_0} \otimes f(\boldsymbol{u}_{i_0}) \otimes \bigoplus_{i \in I_2} c_i \otimes \boldsymbol{u}_i.$$

Hence, we have

$$\begin{aligned} \boldsymbol{v} &= \bigoplus_{i \in I_1} c_i \otimes \boldsymbol{u}_i \oplus \bigoplus_{i \in I_2} c_i \otimes \boldsymbol{u}_i \\ &= \bigoplus_{i \in I_1} c_i \otimes \boldsymbol{u}_i \oplus \bigoplus_{i \in I_2} ((-f(\boldsymbol{v})) \otimes c_{i_0} \otimes c_i) \otimes (f(\boldsymbol{u}_i) \otimes \boldsymbol{u}_{i_0} \oplus f(\boldsymbol{u}_{i_0}) \otimes \boldsymbol{u}_i). \end{aligned}$$

In the other case, there exist two distinct indices $i_1, i_2 \in I_2$ such that $f(\boldsymbol{v}) = c_{i_1} \otimes f(\boldsymbol{u}_{i_1}) = c_{i_2} \otimes f(\boldsymbol{u}_{i_2})$. Then, we similarly compute

$$\begin{aligned} \boldsymbol{v} &= \bigoplus_{i \in I_1} c_i \otimes \boldsymbol{u}_i \oplus \bigoplus_{i \in I_2} c_i \otimes \boldsymbol{u}_i \\ &= \bigoplus_{i \in I_1} c_i \otimes \boldsymbol{u}_i \oplus \bigoplus_{i \in I_{21}} ((-f(\boldsymbol{v})) \otimes c_{i_1} \otimes c_i) \otimes (f(\boldsymbol{u}_i) \otimes \boldsymbol{u}_{i_1} \oplus f(\boldsymbol{u}_{i_1}) \otimes \boldsymbol{u}_i) \\ &\qquad \oplus \bigoplus_{i \in I_{22}} ((-f(\boldsymbol{v})) \otimes c_{i_2} \otimes c_i) \otimes (f(\boldsymbol{u}_i) \otimes \boldsymbol{u}_{i_2} \oplus f(\boldsymbol{u}_{i_2}) \otimes \boldsymbol{u}_i), \end{aligned}$$

where $I_{2p}, p = 1, 2$, are the sets of $i \in I_2$ such that the maximums of $f(\boldsymbol{u}_{i_p})$ and $f(\boldsymbol{u}_i)$ are attained with distinct indices.

Thus, we conclude that (4.6) spans $T(f) \cap U$. □

We remark that tropical linear systems $A \otimes \boldsymbol{x}$ can be expressed by the following two-sided systems:

$$\bigoplus_{j=1}^{n} a_{ij} \otimes x_j = \bigoplus_{j \neq k} a_{ij} \otimes x_j, \quad 1 \leq i \leq m,\ 1 \leq k \leq n.$$

Thus, the proposition above is also proved by using the fact that the solution set of any two-sided system has a finite spanning set [15].

Now, we present our first main result, which characterizes vectors in a basis of the kernel of a max-plus matrix. We start with a characterization of finite vectors in a basis.

Theorem 4.10 ([63]). *Let $A = (a_{ij}) \in \mathbb{R}_{\max}^{m\times n}$, $m \geq n-1$, and $\mathcal{B}$ be a basis of $T(A)$. Then, for each finite vector $\boldsymbol{u} \in \mathcal{B} \cap \mathbb{R}^n$, there exists a subset $I \subset \{1,\dots,m\}$ of size $n-1$ such that $\boldsymbol{u} = c \otimes \boldsymbol{x}^{\mathrm{Cram},A_I}$ for some $c \in \mathbb{R}$. Here, A_I is the $(n-1)\times n$ submatrix of A with rows indexed by I.*

We first reduce the proof of this theorem to a simple case. Suppose $\boldsymbol{u} = {}^t(u_1, u_2, \dots, u_n)$ is a finite vector in the basis $\mathcal{B}$. Without loss of generality, we may assume that $\boldsymbol{u} = {}^t(0,0,\dots,0)$ and the maximum of each row of A is 0. Indeed, if we replace A with $\overline{A}$ by adding u_j to each entry of the jth column of A for $j = 1,2,\dots,n$, then $\boldsymbol{x} \in T(A)$ if and only if $\boldsymbol{x} - \boldsymbol{u} \in T(\overline{A})$. This implies that $\boldsymbol{u}$ is an extreme vector of $T(A)$ if and only if ${}^t(0,0,\dots,0)$ is an extreme vector of $T(\overline{A})$. On the other hand, since

$$\det(\overline{A}_I^{(j)}) = \det(A_I^{(j)}) \otimes \bigotimes_{k\neq j} u_k$$

for any $j = 1,2,\dots,n$ and any subset $I \subset \{1,\dots,m\}$ of size $n-1$, we have

$$\boldsymbol{x}^{\mathrm{Cram},\overline{A}_I} = (u_1 \otimes u_2 \otimes \cdots \otimes u_n) \otimes \left(\boldsymbol{x}^{\mathrm{Cram},A_I} - \boldsymbol{u}\right).$$

Hence, $c \otimes \boldsymbol{x}^{\mathrm{Cram},\overline{A}_I} = {}^t(0,0,\dots,0)$ if and only if $(c \otimes u_1 \otimes u_2 \otimes \cdots \otimes u_n) \otimes \boldsymbol{x}^{\mathrm{Cram},A_I} = \boldsymbol{u}$. Moreover, since the kernel is invariant under the addition of the same scalar to each entry of a single row, we may add $(-\max_j \overline{a}_{ij})$ to each entry of the ith row of $\overline{A} = (\overline{a}_{ij})$ for all $i = 1,2,\dots,m$ so that the maximum of each row becomes 0. Note that the above action multiplies the vector $\boldsymbol{x}^{\mathrm{Cram},\overline{A}_I}$ by a scalar.

Since $\boldsymbol{u} = {}^t(0,0,\dots,0) \in T(A)$, at least two entries are 0 for each row of A. We fix a positive number δ such that $-\delta$ is greater than any negative entry of A. For a subset $J \subset \{1,2,\dots,n\}$, we define the vector $\boldsymbol{v}_J$ by

$$[\boldsymbol{v}_J]_j = \begin{cases} -\delta & \text{if } j \in J, \\ 0 & \text{if } j \notin J. \end{cases}$$

Lemma 4.11. *Let J_1 and J_2 be nonempty subsets of $\{1,2,\dots,n\}$. If $\boldsymbol{v}_{J_p} \in T(A)$ for $p = 1,2$, then $\boldsymbol{v}_{J_1\cap J_2} \in T(A)$.*

Proof. This easily follows from the fact that $T(A)$ is a subspace. □

Proof of Theorem 4.10. We assume that $\boldsymbol{u} = {}^t(0,0,\dots,0) \in \mathcal{B}$ and the maximum of each row of A is 0. Let J be a minimal nonempty subset of $\{1,2,\dots,n\}$ such that $\boldsymbol{v}_J \in T(A)$. Such J always exists since $\boldsymbol{v}_{\{1,2,\dots,n\}} \in T(A)$. Starting with an index $j \in J$, we construct a set I of indices of size $n-1$ such that $\boldsymbol{x}^{\mathrm{Cram},A_I} = {}^t(0,0,\dots,0)$ as follows.

(1) Let $I := \emptyset, I' := \{i \mid a_{ij} = 0\}$ and $K := \{j\}$.

(2) Choose an index $i \in I'$.

(a) If there exists at least one $k \notin K$ such that $a_{ik} = 0$, choose one of such k. Then, we set

$$\begin{aligned} I &:= I \cup \{i\}, \\ I' &:= (I' \cup \{i' \mid a_{i'k} = 0\}) \setminus I, \\ K &:= K \cup \{k\}. \end{aligned}$$

(b) If $a_{ik} \neq 0$ for all $k \notin K$, then we set $I' := I' \setminus \{i\}$.

(3) If we have $I' = \emptyset$, then we finish. Otherwise, we return to (2).

Now we give an intuitive explanation of the above procedure. Starting with a single column j, we extend the column set K connected by 0-edges. A 0-edges means a pair $\{k, k'\}$ where $a_{ik} = a_{ik'} = 0$ for some row i. The set I' is the rows that have at least one 0 in K. Hence, if a row $i \in I'$ has another 0 outside K, we extend K by connecting 0s inside and outside of K. The rows we used in the process are stored in I. However, if all 0s in the row $i \in I'$ are contained in K, such row i will not be used any longer, so it is removed.

First, we show that $K = \{1, 2, \ldots, n\}$ when the procedure is finished. To prove this, let $L = \{1, 2, \ldots, n\} \setminus K$ and show $L = \emptyset$. It follows from the construction of K that one of the following two cases occurs for each row index $i = 1, 2, \ldots, m$: (i) $a_{ik} \neq 0$ for all $k \in K$, (ii) there exist at least two $k \in K$ such that $a_{ik} = 0$. In case (i), there exist $k_1, k_2 \in L$ such that $a_{ik_1} = a_{ik_2} = 0$ and we have

$$\bigoplus_{k=1}^{n} a_{ik} \otimes [\boldsymbol{v}_L]_k = a_{ik_1} \otimes [\boldsymbol{v}_L]_{k_1} = a_{ik_2} \otimes [\boldsymbol{v}_L]_{k_2} = -\delta,$$

because $a_{ik} \otimes [\boldsymbol{v}_L]_k < -\delta$ for $k \notin L$. In case (ii), there exist $k_1, k_2 \notin L$ such that $a_{ik_1} = a_{ik_2} = 0$ and we obtain

$$\bigoplus_{k=1}^{n} a_{ik} \otimes [\boldsymbol{v}_L]_k = a_{ik_1} \otimes [\boldsymbol{v}_L]_{k_1} = a_{ik_2} \otimes [\boldsymbol{v}_L]_{k_2} = 0.$$

In either case, we have $\boldsymbol{v}_L \in T(A)$ and so $\boldsymbol{v}_{J \cap L} \in T(A)$ by Lemma 4.11. Since we have $J \cap L \subsetneq J$ from the condition $j \notin L$, we see that $J \cap L = \emptyset$ by using the minimality of J, which implies $\boldsymbol{v}_J \oplus \boldsymbol{v}_L = {}^t(0, 0, \ldots, 0)$. Since ${}^t(0, 0, \ldots, 0)$ is an extreme vector of $T(A)$, we have $\boldsymbol{v}_J = {}^t(0, 0, \ldots, 0)$ or $\boldsymbol{v}_L = {}^t(0, 0, \ldots, 0)$. Thus, the assumption $J \neq \emptyset$ leads to $L = \emptyset$.

Next, we show that $\boldsymbol{x}^{\mathrm{Cram}, A_I} = {}^t(0, 0, \ldots, 0)$ for the index set I obtained in the above construction. This follows from the fact that we have a bijection $\tau_k : I \to \{1, 2, \ldots, n\} \setminus \{k\}$ such that $a_{i\tau_k(i)} = 0$ for any column k. To make such a bijection, let $p(i), i \in I$, be the column number k when i is last augmented to I' and let $q(i), i \in I$, be the column number k when i is augmented to I. Note that q is a bijection $I \to \{1, 2, \ldots, n\} \setminus \{j\}$, where j is the column we first chose in the procedure. We set $\tau_k(i) := p(i)$ if the row i is in the unique "path" from k to j, and otherwise $\tau_k(i) := q(i)$. Here the "path" is the sequence of rows $(i_1, i_2, \ldots, i_r)$ satisfying $q(i_1) = k, p(i_r) = j$ and $q(i_{l+1}) = p(i_l)$ for $l = 1, 2, \ldots, r-1$, see Figure 4. Now, we see that

$$\bigotimes_{i \in I} a_{i\tau_k(i)} = 0$$

and this appears among the terms of $\det(A_I^{(k)})$. Noting that all entries of $A_I^{(k)}$ are nonpositive, we find $\det(A_I^{(k)}) = 0$. Since this holds for arbitrary column k, we have $\boldsymbol{x}^{\mathrm{Cram}, A_I} = {}^t(0, 0, \ldots, 0)$. This completes the proof of the theorem. $\square$

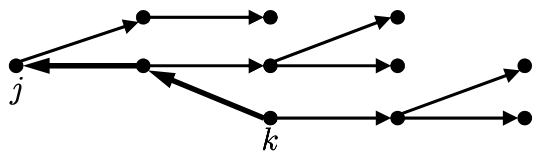

Figure 4: Each vertex represents a column and each edge represents a row $i \in I$. The left and the right endpoints of edges are $p(i)$ and $q(i)$, respectively. The arrows are pointing to $\tau_k(i)$.

Theorem 4.10 is easily extended to vectors that may contain ε.

Corollary 4.12 ([63]). *Let $A = (a_{ij}) \in \mathbb{R}_{\max}^{m \times n}$ and $\mathcal{B}$ be a basis of $T(A)$. We define the matrix $\tilde{A} \in \mathbb{R}_{\max}^{(m+n) \times n}$ by $\begin{pmatrix} A \\ E_n \end{pmatrix}$. Then, for each vector $\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n) \in \mathcal{B}$, there exists a subset $I \subset \{1, \ldots, m+n\}$ of size $n-1$ such that $\boldsymbol{u} = c \otimes \boldsymbol{x}^{\mathrm{Cram}, \tilde{A}_I}$ for some $c \in \mathbb{R}$.*

Proof. Without loss of generality, assume the first d entries of u are finite. Let $\boldsymbol{u}' = {}^t(u_1, u_2 \ldots, u_d)$ and A' be the first d columns of A. Then, $\boldsymbol{u}'$ is an extreme vector of $T(A')$. By Theorem 4.10, there exists a subset $J \subset \{1, \ldots, m\}$ of size $d-1$ such that $\boldsymbol{u}' = c \otimes \boldsymbol{x}^{\mathrm{Cram}, A'_J}$ for some $c \in \mathbb{R}$. Let $I = J \cup \{m+d+1, \ldots, m+n\}$ and consider the matrix

$$\tilde{A}_I = \begin{pmatrix} A'_J & * \\ \mathcal{E} & E_{n-d} \end{pmatrix} \in \mathbb{R}_{\max}^{(n-1) \times n}.$$

If $1 \leq j \leq d$, then

$$\det(\tilde{A}_I^{(j)}) = \det((A')_J^{(j)}) \otimes \det(E_{n-d}) = (-c) \otimes u_j.$$

If $d+1 \leq j \leq n$, then all entries of the $(j-1)$th row of $\tilde{A}_I^{(j)}$ are ε. Hence, $\det(\tilde{A}_I^{(j)}) = \varepsilon$. Thus,

$$\boldsymbol{u} = c \otimes x^{\mathrm{Cram}, \tilde{A}_I}.$$

$\square$

Recall that the dimension of $T(A)$ is the number of vectors in its basis. Surprisingly, it may exceed n for $A \in \mathbb{R}_{\max}^{m \times n}$ though it is a subspace of $\mathbb{R}_{\max}^{n}$. Corollary 4.12 gives an upper bound for the dimension of $T(A)$. It is presented in [6] that a similar bound for the case of solutions of linear inequalities.

Corollary 4.13. *For $A \in \mathbb{R}_{\max}^{m \times n}$, an upper bound for the dimension of $T(A)$ is $\binom{m+n}{n-1}$.*

Example 4.14. Let us consider the matrix

$$A = \begin{pmatrix} 4 & 0 & 5 \\ 1 & 6 & 2 \end{pmatrix}.$$

Applying Cramer's rule for every 2×3 submatrix of

$$\begin{pmatrix} A \\ E_3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 5 \\ 1 & 6 & 2 \\ 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{pmatrix},$$

we obtain the following ten vectors

$$\begin{pmatrix} 11 \\ 6 \\ 10 \end{pmatrix}, \begin{pmatrix} \varepsilon \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ \varepsilon \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ \varepsilon \end{pmatrix}, \begin{pmatrix} \varepsilon \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ \varepsilon \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ \varepsilon \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \end{pmatrix}, \begin{pmatrix} \varepsilon \\ 0 \\ \varepsilon \end{pmatrix}, \begin{pmatrix} \varepsilon \\ \varepsilon \\ 0 \end{pmatrix}.$$

By Corollary 4.12, a basis of $T(A)$ is comprised of some of these vectors. First, we exclude the vectors that are not in $T(A)$, leaving the following candidates:

$$\begin{pmatrix} 11 \\ 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 5 \\ \varepsilon \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ \varepsilon \\ 1 \end{pmatrix}.$$

We can easily verify that these are extreme vectors of $T(A)$ using Proposition 3.6. Since the third vector is a scalar multiple of the second one, a basis of $T(A)$ is

$$\left\{ \begin{pmatrix} 11 \\ 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 5 \\ \varepsilon \\ 4 \end{pmatrix} \right\}.$$

5 Eigenvalue problem

In this section, we summarize results for the max-plus eigenvalue problem presented in, e.g., [9, 13, 42].

5.1 Basic properties

Let $A \in \mathbb{R}_{\max}^{n \times n}$. A scalar $\lambda \in \mathbb{R}_{\max}$ is called an *eigenvalue* of A if there exists a vector $\boldsymbol{u} \in \mathbb{R}_{\max}^{n} \setminus \{\mathcal{E}\}$ such that

$$A \otimes \boldsymbol{u} = \lambda \otimes \boldsymbol{u}.$$

This nontrivial vector $\boldsymbol{u}$ is called an *eigenvector* of A with respect to λ. For $A \in \mathbb{R}_{\max}^{n \times n}$, let $\Lambda(A)$ be the set of all eigenvalues of A. We define the set of vectors

$$U(A, \lambda) = \{\boldsymbol{u} \in \mathbb{R}_{\max}^{n} \mid A \otimes \boldsymbol{u} = \lambda \otimes \boldsymbol{u}\}.$$

Then, we have the following proposition.

Proposition 5.1. *For $A \in \mathbb{R}_{\max}^{n \times n}$ and $c, \lambda \in \mathbb{R}_{\max}$,*

(1) $\lambda \in \Lambda(A) \Longleftrightarrow c \otimes \lambda \in \Lambda(c \otimes A)$,

(2) $U(A, \lambda) \subset U(A^{\otimes k}, \lambda^{\otimes k})$ *for* $k = 1, 2, \ldots,$

(3) $\boldsymbol{u} \in U(A, \lambda) \Longrightarrow c \otimes \boldsymbol{u} \in U(A, \lambda)$,

(4) $\boldsymbol{u}, \boldsymbol{v} \in U(A, \lambda) \Longrightarrow \boldsymbol{u} \oplus \boldsymbol{v} \in U(A, \lambda)$.

From properties (3) and (4), we see that $U(A, \lambda)$ is a subspace of $\mathbb{R}_{\max}^{n}$. We call this subspace the *eigenspace* of A with respect to λ.

5.2 The maximum eigenvalue

The following classical result is fundamental to the max-plus eigenvalue problem.

Proposition 5.2. *Let $A \in \mathbb{R}_{\max}^{n \times n}$. The maximum value of average weights of circuits in $G(A)$ is an eigenvalue of A.*

Proof. Let C be an elementary circuit in $G(A)$ with the maximum average weight, $\lambda = \mathrm{ave}(C)$ and $B = (-\lambda) \otimes A$. Then, the graph $G(B)$ has no circuit with positive weight. If i is a vertex in C, then the ith column of B^*, denoted by $\boldsymbol{\mu}_i$, is identical to that of B^+. Since $B \otimes B^* = B^+$, we have $B \otimes \boldsymbol{\mu}_i = \boldsymbol{\mu}_i$. This implies $A \otimes \boldsymbol{\mu}_i = \lambda \otimes \boldsymbol{\mu}_i$. Thus, λ is an eigenvalue of A and $\boldsymbol{\mu}_i$ is an eigenvector of A with respect to λ. □

Proposition 5.3. *Let $A \in \mathbb{R}_{\max}^{n \times n}$. If there is no circuit in $G(A)$, ε is an eigenvalue of A.*

Proof. Since there is no circuit in $G(A)$, A has a column, say i, whose all entries are ε. Then, the vector $\boldsymbol{e}_i$, whose entries are ε except for the ith entry, is an eigenvector of A with respect to ε. $\square$

Proposition 5.4. *Let $\lambda \neq \varepsilon$ be an eigenvalue of $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$. Then, there is a circuit in $G(A)$ whose average weight is λ.*

Proof. Let $\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n)$ be an eigenvector of A with respect to λ. Take an index i_1 with $u_{i_1} \neq \varepsilon$. Since $A \otimes \boldsymbol{u} = \lambda \otimes \boldsymbol{u}$, there exists an index i_2 such that

$$a_{i_1 i_2} \otimes u_{i_2} = \lambda \otimes u_{i_1}.$$

Since λ and u_{i_1} are finite, $a_{i_1 i_2}$ and u_{i_2} are also finite. Similarly, there exists an index i_3 such that

$$a_{i_2 i_3} \otimes u_{i_3} = \lambda \otimes u_{i_2},$$

where $a_{i_2 i_3}$ and u_{i_3} are finite. In this way, we find $i_4, i_5, \ldots$. Then, we obtain an elementary circuit $C = (i_r, i_{r+1}, \ldots, i_s, i_{s+1} = i_r)$. Summing up the equalities

$$a_{i_k i_{k+1}} \otimes u_{i_{k+1}} = \lambda \otimes u_{i_k}$$

for $k = r, r+1, \ldots, s$, we have

$$\sum_{k=r}^{s} a_{i_k i_{k+1}} + \sum_{k=r}^{s} u_{i_k} = (s-r)\lambda + \sum_{k=r}^{s} u_{i_k}.$$

This leads to the conclusion $\mathrm{ave}(C) = \lambda$. $\square$

For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, if $G(A)$ has a circuit, then let $\lambda(A)$ denote the maximum value of average weights of circuits in $G(A)$; otherwise $\lambda(A) := \varepsilon$. From Proposition 5.2, 5.3 and 5.4, we have the following result.

Theorem 5.5. *The maximum eigenvalue of $A \in \mathbb{R}_{\max}^{n\times n}$ is $\lambda(A)$.*

In the proof of Proposition 5.2 and 5.3, an eigenvector with respect to the maximum eigenvalue is provided. We next characterize a basis of the eigenspace $U(A, \lambda(A))$. Circuits in $G(A)$ with average weights $\lambda(A)$ are called *critical*. The subgraph $G_c(A)$ of $G(A)$ induced by edges in critical circuits is called the *critical graph* of A. The vertex set of the critical graph is denoted by $V_c(A)$. Note that all circuits in the critical graph have the same average weights $\lambda(A)$. In Proposition 5.6, Proposition 5.7 and Theorem 5.8, we denote $-\lambda(A) \otimes A$ by B and the ith column of B^* by $\boldsymbol{\mu}_i$. Note that $\boldsymbol{u} \in U(A, \lambda(A))$ if and only if $\boldsymbol{u} \in U(B, 0)$, and $V_c(A) = V_c(B)$.

Proposition 5.6. *For $A \in \mathbb{R}_{\max}^{n\times n}$, assume that $\lambda(A) \neq \varepsilon$. Then, the set $\{\boldsymbol{\mu}_i \mid i \in V_c(A)\}$ spans the eigenspace $U(A, \lambda(A))$.*

Proof. Suppose $\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n)$ is an eigenvector of A with respect to $\lambda(A)$. We will show that

$$\boldsymbol{u} = \bigoplus_{i \in V_c(A)} u_i \otimes \boldsymbol{\mu}_i.$$

Since the eigenvector $\boldsymbol{u} \in U(B, 0)$ satisfies

$$B^* \otimes \boldsymbol{u} = \boldsymbol{u},$$

the inequality

$$\boldsymbol{u} = \bigoplus_{i=1}^{n} u_i \otimes \boldsymbol{\mu}_i \geq \bigoplus_{i \in V_c(A)} u_i \otimes \boldsymbol{\mu}_i$$

holds. For the reverse inequality, it suffices to prove that there exists a vertex $j \in V_c(A)$ such that

$$u_i \leq u_j \otimes [\boldsymbol{\mu}_j]_i$$

for $i = 1, 2, \ldots, n$. The case $u_i = \varepsilon$ is trivial. Assume that $u_i \neq \varepsilon$ By the iterative use of the equality $B \otimes \boldsymbol{u} = \boldsymbol{u}$, there exists a path $(i = i_0, i_1, \ldots, i_\ell = j), j \in V_c(A)$, such that

$$u_i = \bigotimes_{k=0}^{\ell-1} [B]_{i_k i_{k+1}} \otimes u_j.$$

Indeed, if there were no such path, then there would be a circuit C disjoint from $V_c(A)$ such that $u_i = w(C) \otimes u_i$, which would lead to a contradiction. Since $[\boldsymbol{\mu}_j]_i$ is the maximum weight of all i-j paths, we have

$$\bigotimes_{k=0}^{\ell-1} [B]_{i_k i_{k+1}} \leq [\boldsymbol{\mu}_j]_i,$$

which leads to the desired inequality. □

Proposition 5.7. *For $A \in \mathbb{R}_{\max}^{n \times n}$, assume that $\lambda(A) \neq \varepsilon$. Let i and j are distinct vertices of the critical graph. Then, $\boldsymbol{\mu}_i$ and $\boldsymbol{\mu}_j$ are identical up to scalar if and only if i and j are in the same connected component of the critical graph.*

Proof. Suppose i and j are in the same connected component. For each k, the (k, j) entry of B^* is the maximum weight of all k-j paths in $G(B)$. We may assume that the k-j path with the maximum weight ends on the i-j path with weight $[B^*]_{ij}$ by appending a critical circuit of $G(B)$ passing through i and j to the end of it. This implies $\boldsymbol{\mu}_j = [B^*]_{ij} \otimes \boldsymbol{\mu}_i$.

Conversely, assume that $\boldsymbol{\mu}_j = c \otimes \boldsymbol{\mu}_i$ for some $c \in \mathbb{R}$. Then, we have $[B^*]_{ij} = c \otimes [B^*]_{ii} = c$ and $[B^*]_{ji} = (-c) \otimes [B^*]_{jj} = -c$. Hence, there exists a circuit passing through i and j with weight $c + (-c) = 0$ in $G(B)$. This circuit corresponds to the critical circuit of A. Thus, i and j are in the same connected component of the critical graph. $\square$

Let $N(A)$ be the system of representatives of $V_c(A)$, that is, the set of vertices taken exactly one vertex from each connected component of the critical graph of A.

Theorem 5.8. *For $A \in \mathbb{R}_{\max}^{n \times n}$, assume that $\lambda(A) \neq \varepsilon$. Then, $\{\boldsymbol{\mu}_i \mid i \in N(A)\}$ is a basis of the eigenspace $U(A, \lambda(A))$.*

Proof. From Proposition 5.6 and 5.7, the set $\{\boldsymbol{\mu}_i \mid i \in N(A)\}$ spans the eigenspace. It suffices to show that $\boldsymbol{\mu}_i$ is an extreme vector of $U(A, \lambda(A))$ for $i \in N(A)$. Assume that $\boldsymbol{\mu}_i = \boldsymbol{u} \oplus \boldsymbol{v}$, where $\boldsymbol{u}, \boldsymbol{v} \in U(A, \lambda(A))$. Then, we can express $\boldsymbol{u}$ and $\boldsymbol{v}$ as

$$\boldsymbol{u} = \bigoplus_{i \in N(A)} c_i \otimes \boldsymbol{\mu}_i, \quad \boldsymbol{v} = \bigoplus_{i \in N(A)} d_i \otimes \boldsymbol{\mu}_i, \qquad c_i, d_i \in \mathbb{R}_{\max}.$$

Without loss of generality, there exists a vertex $j \in N(A)$ such that $c_j \otimes [\boldsymbol{\mu}_j]_i = [\boldsymbol{\mu}_i]_i = 0$. Then, we have

$$[\boldsymbol{\mu}_i]_j \geq [\boldsymbol{u}]_j \geq c_j \otimes [\boldsymbol{\mu}_j]_j = c_j.$$

This means that vertices i and j are in the same circuit in $G(B)$ whose weight is greater than or equal to

$$[\boldsymbol{\mu}_j]_i + [\boldsymbol{\mu}_i]_j \geq (-c_j) + c_j = 0.$$

Hence, i and j must be in the same critical circuit of $G(A)$, which implies $i = j$, and hence $\boldsymbol{u} = \boldsymbol{\mu}_i$. Thus, $\boldsymbol{\mu}_i$ is an extreme vector of $U(A, \lambda(A))$. $\square$

We close this subsection by introducing the irreducibility of max-plus square matrices. In a graph G, a vertex j is said to be *reachable* from vertex i if there is an i-j path. A graph is called *strongly connected* if any two vertices are reachable from each other. A matrix $A \in \mathbb{R}_{\max}^{n \times n}$ is *irreducible* if the graph $G(A)$ is strongly connected, and otherwise is *reducible*. We use the convention that 1×1 matrix (ε) is irreducible. Any matrix $A \in \mathbb{R}_{\max}^{n \times n}$ can be rewritten as

$$\begin{pmatrix} A_{1,1} & \mathcal{E} & \cdots & \mathcal{E} \\ A_{2,1} & A_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{E} \\ A_{q,1} & \cdots & A_{q,q-1} & A_{q,q} \end{pmatrix}, \tag{5.1}$$

where $A_{i,i}, i = 1, 2, \ldots, q$, are irreducible, by renumbering the vertices of $G(A)$ so that vertices of $G(A_{j,j})$ are not reachable from those in $G(A_{i,i})$ if $i < j$. The matrix (5.1) is called a *Frobenius normal form* of A. Note that a Frobenius normal form of A is given by $P^{-1} \otimes A \otimes P$ for some permutation matrix P. Hence, we may only consider matrices in Frobenius normal forms. Indeed, if $A \in \mathbb{R}_{\max}^{n \times n}$, λ and $\boldsymbol{u}$ be an eigenvalue and a corresponding eigenvector of A, respectively, and P be a permutation matrix, then

$$(P^{-1} \otimes A \otimes P) \otimes (P^{-1} \otimes \boldsymbol{u}) = P^{-1} \otimes (A \otimes \boldsymbol{u}) = \lambda \otimes (P^{-1} \otimes \boldsymbol{u}).$$

This means that λ is an eigenvalue of $P^{-1} \otimes A \otimes P$ and $P^{-1} \otimes \boldsymbol{u}$ is a corresponding eigenvector.

Proposition 5.9. *If $A = (a_{ij}) \in \mathbb{R}_{\max}^{n \times n}$ is irreducible, then A has exactly one eigenvalue $\lambda(A)$ and corresponding eigenvectors have only finite entries.*

Proof. If ε is an eigenvalue of A, then there is a column in A whose all entries are ε. This contradicts the irreducibility of A. Let $\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n)$ be an eigenvector of A with respect to eigenvalue $\lambda \neq \varepsilon$. Suppose $u_j \neq \varepsilon$. Take an i-j path $(i = i_0, i_1, \ldots, i_\ell = j)$. Then, we have

$$\lambda^{\otimes \ell} \otimes u_i \geq \bigotimes_{k=1}^{\ell} a_{i_{k-1}, i_k} \otimes u_j.$$

This implies $u_i \neq \varepsilon$. Since i-j paths exist for all i by the irreducibility of A, the eigenvector $\boldsymbol{u}$ must be in $\mathbb{R}^n$. Let $(j_0, j_1, \ldots, j_{\ell'} = j_0)$ be a circuit whose average weight is $\lambda(A)$. Then, we have

$$\lambda^{\otimes \ell'} \otimes u_{j_0} \geq \bigotimes_{k=1}^{\ell'} a_{i_{k-1}, i_k} \otimes u_{j_0} = \lambda(A)^{\otimes \ell'} \otimes u_{j_0}.$$

Since the above argument ensures that $u_{j_0} \neq \varepsilon$, we have $\lambda \geq \lambda(A)$. Thus, from Theorem 5.5, we conclude $\lambda = \lambda(A)$. $\square$

Example 5.10. Consider the matrix

$$A = \begin{pmatrix} \varepsilon & -2 & \varepsilon & -5 \\ 0 & -2 & 2 & \varepsilon \\ \varepsilon & \varepsilon & -1 & -6 \\ 1 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

The associated graph $G(A)$ is shown in Figure 5. Since $G(A)$ is strongly connected, A is irreducible and has exactly one eigenvalue $\lambda(A) = -1$. The critical graph of $G(A)$ consists of two circuits $(1, 2, 1)$ and $(3, 3)$. Since

$$(1 \otimes A)^* = \begin{pmatrix} 0 & -1 & 2 & -3 \\ 1 & 0 & 3 & -2 \\ -3 & -4 & 0 & -5 \\ 2 & 1 & 4 & 0 \end{pmatrix},$$

the basis of the eigenspace $U(A,-1)$ is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 4 \end{pmatrix} \right\}.$$

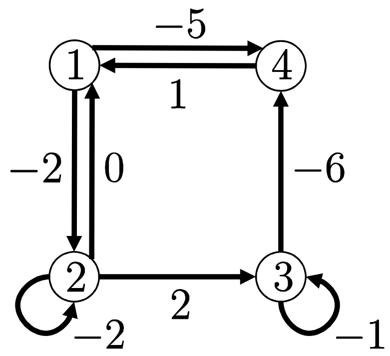

Figure 5: The associated graph for Example 5.10

5.3 All eigenvalues

In this subsection, we focus on all eigenvalues of max-plus matrices. A circuit C is said to be *spectral* if vertices in C are not reachable from any vertices in another circuit C' such that $\mathrm{ave}(C') > \mathrm{ave}(C)$. Then, the set of eigenvalues of a matrix is given as follows.

Theorem 5.11. *Let $A \in \mathbb{R}_{\max}^{n\times n}$. Then, the set of finite eigenvalues is*

$$\{\mathrm{ave}(C) \mid C\text{: spectral circuits in } G(A)\}.$$

Proof. Let λ be the average weight of a spectral circuit. Then, without loss of generality, we assume that A is of the form

$$\begin{pmatrix} A_1 & \mathcal{E} \\ A_2 & A_3 \end{pmatrix},$$

where $\lambda(A_3) = \lambda$. From the results in the previous subsection, there is a vector $\boldsymbol{u} \neq \mathcal{E}$ such that $A_3 \otimes \boldsymbol{u} = \lambda \otimes \boldsymbol{u}$. Hence, we have

$$\begin{pmatrix} A_1 & \mathcal{E} \\ A_2 & A_3 \end{pmatrix} \otimes \begin{pmatrix} \mathcal{E} \\ \boldsymbol{u} \end{pmatrix} = \lambda \otimes \begin{pmatrix} \mathcal{E} \\ \boldsymbol{u} \end{pmatrix}.$$

This leads to the fact that λ is an eigenvalue of A.

Conversely, let $\lambda \neq \varepsilon$ be an eigenvalue of A and $\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n)$ be a corresponding eigenvector. Suppose $u_j \neq \varepsilon$. Then, as in the proof of

Proposition 5.4, λ coincides with the average weight of a circuit C that is reachable from vertex j. Moreover, $u_k \neq \varepsilon$ if k is a vertex in C. On the other hand, as in the proof of Proposition 5.9, λ does not fall below average weights of circuits from which vertices of C are reachable. This proves that C is a spectral circuit in $G(A)$. $\square$

The computation of eigenvectors is straightforward.

Proposition 5.12. *Let $A \in \mathbb{R}_{\max}^{n\times n}$ and $\lambda \neq \varepsilon$ be an eigenvalue of A. Assume that*

$$A = \begin{pmatrix} A_1 & \mathcal{E} \\ A_2 & A_3 \end{pmatrix},$$

where $\lambda(A_1) > \lambda$ and $\lambda(A_3) = \lambda$. If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$ is a basis of the eigenspace $U(A_3, \lambda)$, then

$$\left\{ \begin{pmatrix} \mathcal{E} \\ \boldsymbol{u}_1 \end{pmatrix}, \begin{pmatrix} \mathcal{E} \\ \boldsymbol{u}_2 \end{pmatrix}, \ldots, \begin{pmatrix} \mathcal{E} \\ \boldsymbol{u}_m \end{pmatrix} \right\}$$

is a basis of the eigenspace $U(A, \lambda)$.

Let $U(A)$ be the sum space of all eigenspaces of A, i.e.,

$$U(A) = \left\{ \bigoplus_{\lambda \in \Lambda(A)} \boldsymbol{u}_\lambda \;\middle|\; \boldsymbol{u}_\lambda \in U(A, \lambda) \right\}.$$

Then, we have the following fact.

Corollary 5.13. *Let $A \in \mathbb{R}_{\max}^{n\times n}$ and $\mathcal{B}(A, \lambda)$ be a basis $U(A, \lambda)$ for each eigenvalue λ of A. Then, $\bigcup_{\lambda \in \Lambda(A)} \mathcal{B}(A, \lambda)$ is a basis of $U(A)$. The dimension of $U(A)$ does not exceed n.*

Example 5.14. Consider the matrix

$$A = \begin{pmatrix} \varepsilon & -2 & \varepsilon & \varepsilon \\ 2 & -1 & \varepsilon & \varepsilon \\ \varepsilon & -2 & -3 & \varepsilon \\ -1 & \varepsilon & 0 & -2 \end{pmatrix}.$$

The associated graph $G(A)$ is shown in Figure 6. The spectral circuits in $G(A)$ is $(1, 2, 1)$ with the weight 0 and $(4, 4)$ with the weight -2. Hence, the eigenvalues of A are 0 and -2. The corresponding eigenvectors are ${}^t(0, 2, 0, 0)$ and ${}^t(\varepsilon, \varepsilon, \varepsilon, 0)$, respectively.

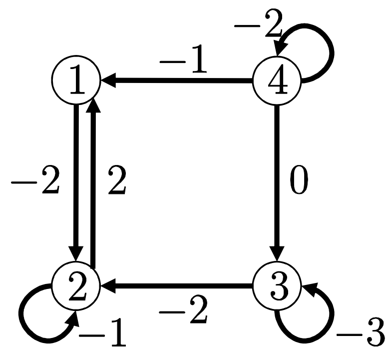

Figure 6: The associated graph for Example 5.14

5.4 Characteristic polynomials

A univariate polynomial in the max-plus algebra is of the form

$$f(t) = c_0 \oplus c_1 \otimes t \oplus c_2 \otimes t^{\otimes 2} \oplus \cdots \oplus c_n \otimes t^{\otimes n}, \quad c_0, c_1, c_2, \ldots, c_n \in \mathbb{R}_{\max}.$$

Max-plus univariate polynomials are piecewise linear functions on $\mathbb{R}_{\max}$. A term $c_k \otimes t^{\otimes k}$ is called *essential* if it contributes to $f(t)$ as a function, that is,

$$c_k \otimes t^{\otimes k} > \bigoplus_{j \neq k} c_j \otimes t^{\otimes j}$$

for some $t \in \mathbb{R}_{\max}$; otherwise, it is called *inessential*. As with the case of standard polynomials over $\mathbb{C}$, each polynomial can be uniquely factorized into the product of linear factors:

$$f(t) = (t \oplus r_1)^{\otimes p_1} \otimes (t \oplus r_2)^{\otimes p_2} \otimes \cdots \otimes (t \oplus r_m)^{\otimes p_m}.$$

Then, r_i and p_i are called a *root* of $f(t)$ and its *multiplicity*, respectively. In the graph of the piecewise linear function $f(t)$, the roots are the bending points of $f(t)$ and the multiplicities are the differences in the slopes of the lines around the roots.

As in the conventional linear algebra, the characteristic polynomial of $A \in \mathbb{R}_{\max}^{n \times n}$ is defined by

$$\varphi_A(t) := \det(A \oplus t \otimes E_n).$$

If we expand the right-hand side, the coefficient of $t^{\otimes k}$ is the maximum weight of multi-circuits in $G(A)$ with length $n - k$. Here, a *multi-circuit* is the set of disjoint elementary circuits in $G(A)$ and its *length* (resp. *weight*) is the sum of the lengths (resp. weights) of all circuits in it. The following factorization algorithm is essentially the same as the operations RESOLUTION and RECTIFY in [24, Section IX], but it is reformulated in terms of graph theory.

Algorithm 5.15. Input: A matrix $A \in \mathbb{R}_{\max}^{n\times n}$
Output: The factorization of the characteristic polynomial of A

(1) Set $i := 0$ and $\mathcal{C}_0 = \emptyset$.

(2) Set $i := i + 1$.

  (a) If there is no multi-circuit in $G(A)$ whose length is larger than $\ell(\mathcal{C}_{i-1})$, then set $m := i$, $\lambda_m := \varepsilon$ and $p_m := n-(p_1+p_2+\cdots+p_{i-1})$ and proceed to (3).

  (b) If there exist multi-circuits in $G(A)$ whose lengths are larger than $\ell(\mathcal{C}_{i-1})$, let $\mathcal{C}_i$ be the multi-circuit $\mathcal{C}$ attaining the maximum value of $\dfrac{w(\mathcal{C}) - w(\mathcal{C}_{i-1})}{\ell(\mathcal{C}) - \ell(\mathcal{C}_{i-1})}$ among them. If there is more than one such multi-circuit, we choose the longest one. We set $\lambda_i := \dfrac{w(\mathcal{C}_i) - w(\mathcal{C}_{i-1})}{\ell(\mathcal{C}_i) - \ell(\mathcal{C}_{i-1})}$ and $p_i := \ell(\mathcal{C}_i) - \ell(\mathcal{C}_{i-1})$, and we repeat (2).

(3) We have the factorization of the characteristic polynomial:

$$\varphi_A(t) = (t \oplus \lambda_1)^{\otimes p_1} \otimes (t \oplus \lambda_2)^{\otimes p_2} \otimes \cdots \otimes (t \oplus \lambda_m)^{\otimes p_m}.$$

We define the *relative average* of multi-circuits $\mathcal{C}'$ with respect to $\mathcal{C}$ by

$$\mathrm{r.ave}(\mathcal{C}, \mathcal{C}') = \begin{cases} \dfrac{w(\mathcal{C}') - w(\mathcal{C})}{\ell(\mathcal{C}') - \ell(\mathcal{C})} & \text{if } \ell(\mathcal{C}') > \ell(\mathcal{C}), \\ \varepsilon & \text{otherwise.} \end{cases}$$

Using this notion, λ_i in Algorithm 5.15 is the maximum value of the relative averages of all multi-circuits with respect to $\mathcal{C}_{i-1}$ in $G(A)$.

As in the conventional linear algebra, the characteristic polynomial of a matrix is related to the eigenvalue problem. A root of the characteristic polynomial of a matrix A is also called an *algebraic eigenvalue* of A [1].

Theorem 5.16 ([23])**.** *For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, the maximum root of its characteristic polynomial is the maximum eigenvalue of A.*

Proof. We observe from the above algorithm that the maximum root of the characteristic polynomial of A is

$$\max_{\mathcal{C}}(\mathrm{r.ave}(\emptyset, \mathcal{C})) = \max_{\mathcal{C}} \frac{w(\mathcal{C})}{\ell(\mathcal{C})} = \max_{C} \mathrm{ave}(C) = \lambda(A),$$

where $\mathcal{C}$ and C are taken over all multi-circuits and all elementary circuits in $G(A)$, respectively. Hence, Theorem 5.5 leads to the conclusion. □

Theorem 5.17 ([1])**.** *All eigenvalues of a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ are roots of its characteristic polynomial.*

Proof. Let λ be an eigenvalue of A. If $\lambda = \lambda(A)$, it is the maximum root of $\varphi_A(t)$ by Theorem 5.16. If $\lambda < \lambda(A)$, we may assume that

$$A = \begin{pmatrix} A_1 & \mathcal{E} \\ A_2 & A_3 \end{pmatrix},$$

where $\lambda(A_3) = \lambda$. By Theorem 5.16, λ is the maximum root of $\varphi_{A_3}(t)$. On the other hand, we have $\varphi_A(t) = \varphi_{A_1}(t) \otimes \varphi_{A_3}(t)$. Hence, λ is a root of $\varphi_A(t)$. $\square$

6 Jordan canonical forms of max-plus matrices

In the previous section, we summarized the theory for the max-plus eigenvalue problem. The results indicate that the number of independent eigenvectors of a matrix does not exceed the number of spectral circuits in the associated graph, which is much smaller than the size of the matrix in general. This means that it is hard to consider the diagonalization of max-plus matrices. In the conventional linear algebra, we can consider a Jordan canonical form of a matrix if it cannot be diagonalized and the columns of the transformation matrix consist of generalized eigenvectors. We would like to imitate this idea in the max-plus algebra, so we first define generalized eigenvectors of max-plus matrices. Then, we propose Jordan canonical forms of matrices together with a necessary and sufficient condition of matrices to have those. This section includes the second main result of this thesis, which is published in [62].

6.1 Generalized eigenspaces

Let $A \in \mathbb{R}_{\max}^{n\times n}$ and λ be an eigenvalue of A. If a vector $\boldsymbol{u} \in \mathbb{R}_{\max}^{n\times n} \setminus \{\mathcal{E}\}$ satisfies $A^{\otimes m} \otimes \boldsymbol{u} = \lambda^{\otimes m} \otimes \boldsymbol{u}$ for some positive integer m, it is called a *generalized eigenvector* of A with respect to λ. Indeed, $(A \oplus \lambda \otimes E_n)^{\otimes (m-1)} \otimes \boldsymbol{u}$ is an eigenvector of A with respect to λ if $A^{\otimes m} \otimes \boldsymbol{u} = \lambda^{\otimes m} \otimes \boldsymbol{u}$. The subspace

$$\tilde{U}(A, \lambda) = \left\{ \boldsymbol{u} \in \mathbb{R}_{\max}^{n} \mid A^{\otimes m} \otimes \boldsymbol{u} = \lambda^{\otimes m} \otimes \boldsymbol{u} \text{ for some } m \geq 1 \right\}.$$

is called the *generalized eigenspace* of A with respect to λ. The sum of all generalized eigenspaces is denoted by $\tilde{U}(A)$, i.e.,

$$\tilde{U}(A) = \left\{ \bigoplus_{\lambda \in \Lambda(A)} \boldsymbol{u}_\lambda \;\middle|\; \boldsymbol{u}_\lambda \in \tilde{U}(A, \lambda) \right\}.$$

From the following proposition and Corollary 5.13, we see that a basis of $\tilde{U}(A)$ is the union of bases of $\tilde{U}(A, \lambda)$ for $\lambda \in \Lambda(A)$ and the dimension of $\tilde{U}(A)$ does not exceed n.

Proposition 6.1. *Let $A \in \mathbb{R}_{\max}^{n\times n}$ and L be a common multiple of the lengths of all elementary circuits in $G(A)$ with average weights $\lambda(A)$. Then,*

$$\tilde{U}(A, \lambda(A)) = U(A^{\otimes L}, \lambda(A)^{\otimes L}).$$

Proof. The inclusion $U(A^{\otimes L}, \lambda(A)^{\otimes L}) \subset \tilde{U}(A, \lambda)$ is obvious. To prove the opposite inclusion, suppose $\boldsymbol{u} \in \mathbb{R}_{\max}^{n}$ satisfies $A^{\otimes m} \otimes \boldsymbol{u} = \lambda^{\otimes m} \otimes \boldsymbol{u}$ for some positive integer m. Then, we have

$$A^{\otimes Lm} \otimes \boldsymbol{u} = A^{\otimes (L-1)m} \otimes (\lambda(A)^{\otimes m} \otimes \boldsymbol{u}) = \cdots = \lambda(A)^{\otimes Lm} \otimes \boldsymbol{u}.$$

So, it suffices to show that $U(A^{\otimes Lm}, \lambda(A)^{\otimes Lm}) = U(A^{\otimes L}, \lambda(A)^{\otimes L})$. First, we see that

$$\left((-\lambda(A) \otimes A)^{\otimes Lm}\right)^* \leq \left((-\lambda(A) \otimes A)^{\otimes L}\right)^* \tag{6.1}$$

On the other hand, fix a vertex i in a critical circuit of $G(A)$, or equivalently, $G(A^{\otimes L})$ or $G(A^{\otimes Lm})$. Then, for any j-i path in $G(-\lambda(A) \otimes A)$ whose length is a multiple of L, we can construct a j-i path with the same weight whose length is a multiple of Lm. This is because of the fact that there is a path with weight 0 and length L that passes through i. Hence, the ith column of both sides of (6.1) are identical. Now, Proposition 5.6 yields $U(A^{\otimes Lm}, \lambda(A)^{\otimes Lm}) = U(A^{\otimes L}, \lambda(A)^{\otimes L})$. $\square$

6.2 Jordan canonical forms

For $\lambda \neq \varepsilon$ and an integer $m \geq 1$, we define a matrix of the form

$$J(\lambda, m) = \begin{pmatrix} \varepsilon & \lambda & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \varepsilon & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \varepsilon \\ \varepsilon & \ddots & \ddots & \ddots & \lambda \\ \lambda & \varepsilon & \cdots & \varepsilon & \varepsilon \end{pmatrix} \in \mathbb{R}_{\max}^{m \times m}.$$

In this section, we consider the problem when a max-plus matrix $A \in \mathbb{R}_{\max}^{n \times n}$ is transformed into the matrix of the form

$$J = \begin{pmatrix} J(\lambda_1, m_1) & \mathcal{E} & \cdots & \mathcal{E} \\ \mathcal{E} & J(\lambda_2, m_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{E} \\ \mathcal{E} & \cdots & \mathcal{E} & J(\lambda_r, m_r) \end{pmatrix}. \tag{6.2}$$

If there exists a nonsingular matrix $P \in \mathbb{R}_{\max}^{n \times n}$ such that $A \otimes P = P \otimes J$, then we call J a *Jordan canonical form* of A. This name comes from the following observation. Let $P_i \in \mathbb{R}_{\max}^{n \times m_i}$ be the columns of P corresponding to the block $J(\lambda_i, m_i)$. Then, we have

$$\begin{aligned} A^{\otimes m_i} \otimes P_i = A^{\otimes m_i - 1} \otimes (P_i \otimes J(\lambda_i, m_i)) = \cdots =& P_i \otimes J(\lambda_i, m_i)^{\otimes m_i} \\ =& \lambda_i^{\otimes m_i} \otimes P_i. \end{aligned}$$

This leads to the fact that all columns of P are generalized eigenvectors of A with respect to λ_i, which is analogous to the conventional case.

Remark 6.2. For a max-plus matrix A, we must note that the inclusion

$$(\text{the column space of } P) \subset \tilde{U}(A). \tag{6.3}$$

may be strict even if P is a nonsingular matrix satisfying $A \otimes P = P \otimes J$, where J is a Jordan canonical form of A. To make a similar result to the conventional theory, a transformation matrix P achieving the equality in (6.3) seems the best choice. Since a basis of a max-plus linear space is unique up to scalar multiples by Theorem 3.5, the equality holds if and only if the columns of P form a basis of $\tilde{U}(A)$.

We present the statement of the second main result.

Theorem 6.3 ([62]). *A matrix $A \in \mathbb{R}_{\max}^{n \times n}$ has a Jordan canonical form if and only if each vertex in $G(A)$ is contained in exactly one spectral circuit. In this case, we can choose the transformation matrix P so that the equality in* (6.3) *holds.*

To prove "if part" of the theorem, we first show the following lemma.

Lemma 6.4. *For a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n \times n}$, let $C = (i_0, i_1, i_2, \ldots, i_\ell = i_0)$ be a spectral circuit of $G(A)$ and λ be the average weight of C. If L is a multiple of $\ell = \ell(C)$, then for $k = 1, 2, \ldots, \ell$, we have*

$$A \otimes \boldsymbol{\mu}_{i_k} = a_{i_{k-1} i_k} \otimes \boldsymbol{\mu}_{i_{k-1}},$$

where $\boldsymbol{\mu}_i$ is the ith column of $\left((-\lambda \otimes A)^{\otimes L}\right)^*$.

Proof. The jth entry of vector $\boldsymbol{\mu}_i$ represents the maximum value of weights of all j-i paths in $G(-\lambda \otimes A)$ whose lengths are multiples of L. Since C is spectral and $w(C) = 0$ in $G(-\lambda \otimes A)$, $\boldsymbol{\mu}_{i_k}$ does not contain ∞ for $k = 1, 2, \ldots, \ell$. Consider the vector $(-\lambda \otimes A) \otimes \boldsymbol{\mu}_{i_k}$. The j th entry of that vector represents the maximum value of weights of all j-i_k paths in $G(-\lambda \otimes A)$ whose lengths are congruent to 1 modulo L. Let P be such a j-i_k path with the maximum weight. Then, we have

$$w(P) = \left[(-\lambda \otimes A) \otimes \boldsymbol{\mu}_{i_k}\right]_j.$$

On the other hand, we may assume that the last edge of P is (i_{k-1}, i_k) by appending L/ℓ copies of C to the end of the path. Then, P is the concatenation of the j-i_{k-1} path P' and the edge (i_{k-1}, i_k). The length of P' is a multiple of L and it has the maximum weight among all j-i_{k-1} paths whose lengths are multiples of L. As the weight of the edge (i_{k-1}, i_k) in $G(-\lambda \otimes A)$ is $-\lambda \otimes a_{i_{k-1} i_k}$, we have

$$w(P) = [\boldsymbol{\mu}_{i_{k-1}}]_j \otimes (-\lambda \otimes a_{i_{k-1} i_k}).$$

Hence, we obtain

$$\left[(-\lambda \otimes A) \otimes \boldsymbol{\mu}_{i_k}\right]_j = [\boldsymbol{\mu}_{i_{k-1}}]_j \otimes (-\lambda \otimes a_{i_{k-1} i_k}).$$

As this holds for $j = 1, 2, \ldots, n$, we have

$$A \otimes \boldsymbol{\mu}_{i_k} = a_{i_{k-1} i_k} \otimes \boldsymbol{\mu}_{i_{k-1}}$$

by multiplying λ to both sides in the sense of max-plus arithmetic. $\square$

Proof ("if" part of Theorem 6.3). Let $\lambda_1 > \lambda_2 > \cdots > \lambda_q$ be the eigenvalues of A. Renumbering the vertices of $G(A)$, if necessary, we assume that

$$A = \begin{pmatrix} A_{1,1} & \mathcal{E} & \cdots & \mathcal{E} \\ A_{2,1} & A_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{E} \\ A_{q,1} & \cdots & A_{q,q-1} & A_{q,q} \end{pmatrix},$$

where the vertices of $G(A_{i,i})$ are covered by the spectral circuits of average weights λ_i without duplications. We remark that the upper right blocks of A are $\mathcal{E}$ since otherwise some of λ_i would not be an eigenvalue of A. We will prove the existence of a Jordan canonical form of A by induction on q.

First, we assume that $q = 1$, which means $G(A)$ has exactly one eigenvalue λ. Let $\{C_1, C_2, \ldots, C_r\}$ be the set of all spectral circuits, where $C_j = (i_{j,0}, i_{j,1}, i_{j,2}, \ldots, i_{j,\ell(C_j)} = i_{j,0})$ for $j = 1, 2, \ldots, r$. Note that $\ell(C_1) + \ell(C_2) + \cdots + \ell(C_r) = n$ since each vertex is contained in exactly one C_i. Let L be the least common multiple of $\ell(C_1), \ell(C_2), \ldots, \ell(C_r)$ and $\boldsymbol{\mu}_i$ be the ith column of $\big((-\lambda \otimes A)^{\otimes L}\big)^*$. By Lemma 6.4, we have

$$A \otimes \boldsymbol{\mu}_{i_{j,k}} = a_{i_{j,k-1}, i_{j,k}} \otimes \boldsymbol{\mu}_{i_{j,k-1}}$$

for $j = 1, 2, \ldots, r$ and $k = 1, 2, \ldots, \ell(C_j)$. Hence, if we replace $\boldsymbol{\mu}_{i_{j,k}}$ with

$$\boldsymbol{\nu}_{i_{j,k}} = \lambda^{\otimes k} \otimes \left(- \bigotimes_{s=1}^{k} a_{i_{j,s-1}, i_{j,s}} \right) \otimes \boldsymbol{\mu}_{i_{j,k}},$$

we have

$$A \otimes \boldsymbol{\nu}_{i_{j,k}} = \lambda \otimes \boldsymbol{\nu}_{i_{j,k-1}}.$$

By setting

$$P = \left(\boldsymbol{\nu}_{i_{1,1}}, \boldsymbol{\nu}_{i_{1,2}}, \ldots, \boldsymbol{\nu}_{i_{1,\ell(C_1)}}, \ldots\ldots, \boldsymbol{\nu}_{i_{r,1}}, \boldsymbol{\nu}_{i_{r,2}}, \ldots, \boldsymbol{\nu}_{i_{r,\ell(C_r)}} \right) \in \mathbb{R}_{\max}^{n \times n},$$

we obtain

$$A \otimes P = P \otimes \begin{pmatrix} J(\lambda, \ell(C_1)) & \mathcal{E} & \cdots & \mathcal{E} \\ \mathcal{E} & J(\lambda, \ell(C_2)) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{E} \\ \mathcal{E} & \cdots & \mathcal{E} & J(\lambda, \ell(C_r)) \end{pmatrix}.$$

To prove

$$\begin{pmatrix} J(\lambda, \ell(C_1)) & \mathcal{E} & \cdots & \mathcal{E} \\ \mathcal{E} & J(\lambda, \ell(C_2)) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{E} \\ \mathcal{E} & \cdots & \mathcal{E} & J(\lambda, \ell(C_r)) \end{pmatrix}$$

is a Jordan canonical form of A, it remains to show that P is nonsingular. Since P is obtained from $\left((-\lambda \otimes A)^{\otimes L}\right)^*$ by permuting the columns and multiplying the columns by scalars, P is nonsingular if and only if $\left((-\lambda \otimes A)^{\otimes L}\right)^*$ is nonsingular. If $\left((-\lambda \otimes A)^{\otimes L}\right)^*$ were singular, the maximum in det $\left((-\lambda \otimes A)^{\otimes L}\right)^*$ would be attained at least twice: with the identity permutation and another permutation. This implies that the graph $G\left((-\lambda \otimes A)^{\otimes L}\right)$ has a circuit with weight 0 besides loops. Since L is the least common multiple of $\ell(C_1), \ell(C_2), \ldots, \ell(C_r)$, it occurs only if some of two spectral circuits share a vertex, which contradicts the assumption of the theorem.

Next, we assume that the assertion is proved for matrices that have $q-1$ different eigenvalues. For the block $A_{1,1} \in \mathbb{R}_{\max}^{d_1 \times d_1}$, let $\{C_1, C_2, \ldots, C_r\}$ be the set of spectral circuits covering $\{1, 2, \ldots, d_1\}$, where

$$C_j = (i_{j,0}, i_{j,1}, i_{j,2}, \ldots, i_{j,\ell(C_j)} = i_{j,0}), \quad j = 1, 2, \ldots, r.$$

Recall that $\operatorname{ave}(C_i) = \lambda_1$ for all $i = 1, \ldots, r$. Let L be the least common multiple of $\ell(C_1), \ell(C_2), \ldots, \ell(C_r)$ and $\boldsymbol{\mu}_i$ be the ith column of $\left((-\lambda_1 \otimes A)^{\otimes L}\right)^*$. As above, setting

$$\boldsymbol{\nu}_{i_{j,k}} = \lambda_1^{\otimes k} \otimes \left(- \bigotimes_{s=1}^{k} a_{i_{j,s-1}, i_{j,s}} \right) \otimes \boldsymbol{\mu}_{i_{j,k}}$$

and

$$P_1 = \left(\boldsymbol{\nu}_{i_{1,1}}, \boldsymbol{\nu}_{i_{1,2}}, \ldots, \boldsymbol{\nu}_{i_{1,\ell(C_1)}}, \ldots\ldots, \boldsymbol{\nu}_{i_{r,1}}, \boldsymbol{\nu}_{i_{r,2}}, \ldots, \boldsymbol{\nu}_{i_{r,\ell(C_r)}} \right) \in \mathbb{R}_{\max}^{n \times d_1},$$

we get

$$A \otimes P_1 = P_1 \otimes J_1,$$

where

$$J_1 = \begin{pmatrix} J(\lambda_1, \ell(C_1)) & \mathcal{E} & \cdots & \mathcal{E} \\ \mathcal{E} & J(\lambda_1, \ell(C_2)) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{E} \\ \mathcal{E} & \cdots & \mathcal{E} & J(\lambda_1, \ell(C_m)) \end{pmatrix} \in \mathbb{R}_{\max}^{d_1 \times d_1}.$$

For $(n - d_1) \times (n - d_1)$ matrix

$$A' = \begin{pmatrix} A_{2,2} & \mathcal{E} & \cdots & \mathcal{E} \\ A_{3,2} & A_{3,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{E} \\ A_{q,2} & \cdots & A_{q,q-1} & A_{q,q} \end{pmatrix},$$

by induction, there exists a nonsingular matrix P' and a Jordan canonical form J' satisfying $A' \otimes P' = P' \otimes J'$. Thus, we have

$$A \otimes \left(P_1 \;\middle|\; \begin{matrix} \mathcal{E} \\ P' \end{matrix} \right) = \left(P_1 \;\middle|\; \begin{matrix} \mathcal{E} \\ P' \end{matrix} \right) \otimes \begin{pmatrix} J_1 & \mathcal{E} \\ \mathcal{E} & J' \end{pmatrix}.$$

Here $P = \left(P_1 \;\middle|\; \begin{matrix} \mathcal{E} \\ P' \end{matrix} \right)$ is nonsingular. Indeed, since both P' and the first d_1 rows of P_1 are nonsingular, the maximum in $\det P$ is attained precisely once. Thus, we conclude that $\begin{pmatrix} J_1 & \mathcal{E} \\ \mathcal{E} & J' \end{pmatrix}$ is a Jordan canonical form of A.

The last statement of the theorem follows from the fact that the transformation matrix P above achieves the equality in (6.3), which can be verified by inductively use of Proposition 6.1. □

For the proof of the "only if" part, let $G(A)[\lambda]$ denote the subgraph of $G(A)$ induced by all spectral circuits with average weights λ. Clearly, $G(A)[\lambda(A)]$ is the critical graph $G_c(A)$ defined in Section 5.

Proof ("only if"part of Theorem 6.3). Assume that $A \otimes P = P \otimes J$, where P is a nonsingular matrix and J is a Jordan canonical form of A. We take L as the least common multiple of the sizes of all diagonal blocks in J. Let d_λ be the sum of the sizes of the diagonal blocks in J whose eigenvalues are equal to λ, and let $P_\lambda \in \mathbb{R}_{\max}^{n \times d_\lambda}$ be the collection of the corresponding columns of P. For each eigenvalue λ of A, all columns of P_λ are in $\tilde{U}(A, \lambda)$. The dimension of $\tilde{U}(A, \lambda)$ is at least d_λ; otherwise the column rank of P_λ would be less than d_λ and hence the column rank of P would be less than n. Then, Lemma 3.7 implies P is singular, which is a contradiction. By Theorem 5.8 and Proposition 6.1, we will find out that the subgraph $G(A^{\otimes L})[\lambda^{\otimes L}]$ has at least d_λ connected components. Since this holds for all eigenvalues λ of J, the subgraph $G(A^{\otimes L})[\lambda^{\otimes L}]$ must be a collection of d_λ loops and for each vertex j there exists exactly one λ such that $G(A^{\otimes L})[\lambda^{\otimes L}]$ contains loop (j, j). This means that vertex j is contained in a spectral circuit of $G(A)$ with average weight λ. We will finish the proof with the following lemma. □

Lemma 6.5. *For a matrix $A \in \mathbb{R}_{\max}^{n \times n}$ and an eigenvalue $\lambda \neq \varepsilon$, suppose that there exists an integer L such that $G(A^{\otimes L})[\lambda^{\otimes L}]$ is a collection of loops. Then, any two spectral circuits of $G(A)$ with average weight λ have no common vertices.*

Proof. On the contrary, suppose that there exists a vertex i that is contained in two circuits C and C' of $G(A)[\lambda]$. Without loss of generality, we may assume that the predecessor of i in C, namely j, is different from that in C', namely j'. Then, $G(A)[\lambda]$ has a j'-j path whose length is a multiple of L. Indeed, we can find such a path as a concatenation of the edge (j', i), $(L-1)$

copies of C_i and i-j path along C_i. Similarly, $G(A)[\lambda]$ also has a j-j' path whose length is a multiple of L. Hence, $G(A^{\otimes L})[\lambda^{\otimes L}]$ has both j'-j path and j-j' path. This contradicts the fact that $G(A^{\otimes L})[\lambda^{\otimes L}]$ is a collection of loops. $\square$

The proof of the "if part" of Theorem 6.3 indicates how to compute a Jordan canonical form and a transformation matrix. We will show this by an example.

Example 6.6. Consider the matrix

$$A = \left(\begin{array}{ccccc|cc} \varepsilon & 4 & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 3 & \varepsilon & 0 & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 1 & 3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & 1 & \varepsilon \end{array}\right).$$

The spectral circuits of $G(A)$ are $(1,2)$, $(3,4,5)$ and $(6,7)$, whose average weights are 4, 4 and 2, respectively. Since $G(A)$ satisfies the condition in Theorem 6.3, A has a Jordan canonical form. Now, we will compute that. First, we focus on the spectral circuits with average weight 4. Since the least common multiple of 2 and 3 is 6, we compute

$$\left(((-4) \otimes A)^{\otimes 6}\right)^* = \begin{pmatrix} 0 & -2 & -1 & 0 & 0 & \varepsilon & \varepsilon \\ -2 & 0 & -1 & 0 & 0 & \varepsilon & \varepsilon \\ -1 & -1 & 0 & -1 & -1 & \varepsilon & \varepsilon \\ -2 & -2 & -3 & 0 & -2 & \varepsilon & \varepsilon \\ -2 & -2 & -3 & -2 & 0 & \varepsilon & \varepsilon \\ -5 & -5 & -4 & -4 & -5 & 0 & -12 \\ -4 & -4 & -5 & -2 & -4 & -14 & 0 \end{pmatrix}.$$

Setting

$$\boldsymbol{\mu}_1 = {}^t(0,-2,-1,-2,-2,-5,-4), \quad \boldsymbol{\mu}_2 = {}^t(-2,0,-1,-2,-2,-5,-4),$$
$$\boldsymbol{\mu}_3 = {}^t(-1,-1,0,-3,-3,-4,-5), \quad \boldsymbol{\mu}_4 = {}^t(0,0,-1,0,-2,-4,-2),$$
$$\boldsymbol{\mu}_5 = {}^t(0,0,-1,-2,0,-5,-4),$$

we have

$$A \otimes \boldsymbol{\mu}_1 = 4 \otimes \boldsymbol{\mu}_2, \ A \otimes \boldsymbol{\mu}_2 = 4 \otimes \boldsymbol{\mu}_1,$$
$$A \otimes \boldsymbol{\mu}_3 = 3 \otimes \boldsymbol{\mu}_5, \ A \otimes \boldsymbol{\mu}_4 = 5 \otimes \boldsymbol{\mu}_3, \ A \otimes \boldsymbol{\mu}_5 = 4 \otimes \boldsymbol{\mu}_4.$$

Similarly, since

$$\left(\left((-2) \otimes \begin{pmatrix} 1 & 3 \\ 1 & \varepsilon \end{pmatrix}\right)^{\otimes 2}\right)^* = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix},$$

we set

$$\boldsymbol{\mu}_6 = {}^t(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, 0, -2), \quad \boldsymbol{\mu}_7 = {}^t(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, 0, 0),$$

and we have

$$A \otimes \boldsymbol{\mu}_6 = -4 \otimes \boldsymbol{\mu}_7, \ A \otimes \boldsymbol{\mu}_7 = -2 \otimes \boldsymbol{\mu}_6.$$

Hence, replacing $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_7$ with

$$\begin{aligned}
&\boldsymbol{\nu}_1 = (4-4) \otimes \boldsymbol{\mu}_1, \quad \boldsymbol{\nu}_2 = \boldsymbol{\mu}_2, \\
&\boldsymbol{\nu}_3 = (4-3) \otimes \boldsymbol{\mu}_3, \quad \boldsymbol{\nu}_4 = (4^{\otimes 2} - (3 \otimes 5)) \otimes \boldsymbol{\mu}_4, \quad \boldsymbol{\nu}_5 = \boldsymbol{\mu}_5, \\
&\boldsymbol{\nu}_6 = (2-1) \otimes \boldsymbol{\mu}_6, \quad \boldsymbol{\nu}_7 = \boldsymbol{\mu}_7,
\end{aligned}$$

and setting $P = \begin{pmatrix} \boldsymbol{\nu}_1 & \boldsymbol{\nu}_2 & \boldsymbol{\nu}_3 & \boldsymbol{\nu}_4 & \boldsymbol{\nu}_5 & \boldsymbol{\nu}_6 & \boldsymbol{\nu}_7 \end{pmatrix}$, we have

$$A \otimes P = P \otimes \left(\begin{array}{cc|ccc|cc}
\varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\hline
\varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\hline
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon
\end{array}\right).$$

Further, $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3, \boldsymbol{\nu}_4, \boldsymbol{\nu}_5\}$ and $\{\boldsymbol{\nu}_6, \boldsymbol{\nu}_7\}$ are bases of generalized eigenspaces $\tilde{U}(A, 4)$ and $\tilde{U}(A, 2)$, respectively.

7 Algebraic eigenvectors associated with roots of max-plus characteristic polynomials

As we saw in Section 5, the maximum root of the characteristic polynomial of a matrix is its eigenvalue. However, other roots may not be eigenvalues. Thus, our concern is to clarify the roles of the roots of the characteristic polynomial that are not maximums. To investigate this problem, we introduce the notion of algebraic eigenvectors associated with roots of the characteristic polynomial of a matrix. Then, we show some properties of algebraic eigenvectors analogous to those of the conventional eigenvectors. The result in this section is presented in the author's publication [64].

7.1 One assumption for generic matrices

For $A \in \mathbb{R}_{\max}^{n\times n}$ and variable t, we define $2n \times 2n$ matrix

$$\tilde{A}(t) = \begin{pmatrix} A & t \otimes E \\ E & E \end{pmatrix}.$$

We note that $\varphi_A(t) = \det \tilde{A}(t)$ as the functions of t. The matrix $\tilde{A}(t)$ also admits a graph theoretical characterization. We say that a permutation $\pi \in S_{2n}$ is *finite* with respect to $\tilde{A}(t) = (\tilde{a}_{ij})$ if $\tilde{a}_{i\pi(i)} \neq \varepsilon$ for $i = 1, 2, \ldots, 2n$. For a multi-circuit $\mathcal{C}$ in $G(A)$, we define a finite permutation $\pi^{\mathcal{C}} \in S_{2n}$ as follows:

$$\pi^{\mathcal{C}}(i) = \begin{cases} (\text{the next vertex of } i \text{ in } \mathcal{C}) & \text{if } i \in V(\mathcal{C}), 1 \leq i \leq n, \\ i+n & \text{if } i \notin V(\mathcal{C}), 1 \leq i \leq n, \\ i & \text{if } i-n \in V(\mathcal{C}), n+1 \leq i \leq 2n, \\ i-n & \text{if } i-n \notin V(\mathcal{C}), n+1 \leq i \leq 2n. \end{cases}$$

The map $\mathcal{C} \mapsto \pi^{\mathcal{C}}$ gives a one to one correspondence between multi-circuits in $G(A)$ and finite permutations with respect to $\tilde{A}(t)$. For $\lambda \neq \varepsilon$, we say that a multi-circuit $\mathcal{C}$ is λ-*maximal* if $\pi^{\mathcal{C}}$ attains the maximum of $\det \tilde{A}(\lambda)$. A multi-circuit $\mathcal{C}$ is ε-*maximal* if $\pi^{\mathcal{C}}$ attains the maximum of $\det \tilde{A}(\overline{\lambda})$ for a sufficiently small finite value $\overline{\lambda}$. Note that the ε-maximal multi-circuit has the maximum length among all multi-circuits in $G(A)$. In Algorithm 5.15, both $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$ are λ_i-maximal multi-circuits.

Lemma 7.1. *If λ is a root of the characteristic polynomial $\varphi_A(t)$, then the matrix $\tilde{A}(\lambda)$ is singular.*

Proof. If $\lambda = \varepsilon$ is a root of $\varphi_A(t)$, the graph $G(A)$ has no multi-circuit with length n. Then, both $\det A$ and $\det \tilde{A}(\varepsilon)$ are ε, which means that $\tilde{A}(\varepsilon)$ is singular.

If $\lambda \neq \varepsilon$ is a root of $\varphi_A(t)$, there exist at least two terms, say $c_{k_1} \otimes t^{\otimes k_1}$ and $c_{k_2} \otimes t^{\otimes k_2}$, $k_1 \neq k_2$, such that

$$c_{k_1} \otimes \lambda^{\otimes k_1} = c_{k_2} \otimes \lambda^{\otimes k_2} = \varphi_A(\lambda) = \det \tilde{A}(\lambda).$$

Both $c_{k_1} \otimes \lambda^{\otimes k_1}$ and $c_{k_2} \otimes \lambda^{\otimes k_2}$ appear in the summands of $\det \tilde{A}(\lambda)$. Hence, $\tilde{A}(\lambda)$ is singular. □

Generally, the converse of the above lemma is not true. However, it holds under the following assumption, which is so weak that it is satisfied by generic matrices.

Assumption 7.2. For a matrix $A \in \mathbb{R}_{\max}^{n \times n}$, we assume that all essential terms of its characteristic polynomial are attained with exactly one permutation. Equivalently, if $c_k \otimes t^{\otimes k}$ is an essential term of $\varphi_A(t)$, there exists exactly one multi-circuit $\mathcal{C}$ with $\ell(\mathcal{C}) = n - k$ and $w(\mathcal{C}) = c_k$ in $G(A)$.

Proposition 7.3. *Under Assumption 7.2 for a matrix $A \in \mathbb{R}_{\max}^{n \times n}$, λ is a root of the characteristic polynomial $\varphi_A(t)$ if and only if the matrix $\tilde{A}(\lambda)$ is singular.*

Proof. The "only if" part has been proved in Lemma 7.1. For the "if" part, suppose that $\tilde{A}(\lambda)$ is singular. If the maximum of $\varphi_A(\lambda)$ is attained with exactly one term, say $c_k \otimes \lambda^{\otimes k}$, then $c_k \otimes t^{\otimes k}$ must be an essential term. From Assumption 7.2, the maximum of $\det \tilde{A}(\lambda)$ is also attained exactly once, which leads to a contradiction. Thus, the maximum of $\varphi_A(\lambda)$ is attained at least twice. Hence, λ is a root of $\varphi_A(t)$. □

In terms of graph theory, λ is a finite root of $\varphi_A(t)$ if and only if there exist at least two λ-maximal multi-circuits with different lengths in the associated graph $G(A)$. Hereinafter, we proceed with our argument under Assumption 7.2. We note that this kind of assumption also appears in the literature on the supertropical algebra [48].

7.2 Definition of algebraic eigenvectors

For a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n \times n}$ and a multi-circuit $\mathcal{C}$ in $G(A)$, we define four types of matrices, $A_{\mathcal{C}}, A_{\setminus\mathcal{C}}, E_{\mathcal{C}}$ and $E_{\setminus\mathcal{C}}$ as follows:

$$[A_{\mathcal{C}}]_{ij} = \begin{cases} a_{ij} & \text{if } (i,j) \in E(\mathcal{C}), \\ \varepsilon & \text{otherwise,} \end{cases} \qquad [A_{\setminus\mathcal{C}}]_{ij} = \begin{cases} \varepsilon & \text{if } (i,j) \in E(\mathcal{C}), \\ a_{ij} & \text{otherwise,} \end{cases}$$

$$[E_{\mathcal{C}}]_{ij} = \begin{cases} e & \text{if } i = j,\ i \in V(\mathcal{C}), \\ \varepsilon & \text{otherwise,} \end{cases} \qquad [E_{\setminus\mathcal{C}}]_{ij} = \begin{cases} e & \text{if } i = j,\ i \notin V(\mathcal{C}), \\ \varepsilon & \text{otherwise.} \end{cases}$$

Here $V(\mathcal{C})$ and $E(\mathcal{C})$ denote the vertex set and the edge set of $\mathcal{C}$, respectively. Now we present the main result of [64] together with the definition of algebraic eigenvectors.

Theorem 7.4 ([64])**.** *Let $A \in \mathbb{R}_{\max}^{n \times n}$. Then, $\lambda \in \mathbb{R}_{\max}$ is an algebraic eigenvalue, i.e., a root of $\varphi_A(t)$, if and only if there exists a λ-maximal multi-circuit $\mathcal{C}$ and a vector $\boldsymbol{u} \neq \mathcal{E}$ such that*

$$(A_{\setminus\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{u} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus\mathcal{C}}) \otimes \boldsymbol{u}.$$

We call such a nontrivial vector $\boldsymbol{u}$ an algebraic eigenvector of A with respect to λ.

Remark 7.5. If λ is the maximum eigenvalue of A, then it coincides with the maximum algebraic eigenvalue and hence $\mathcal{C} = \emptyset$ is λ-maximal. Thus, the equation in Theorem 7.4 will be $A \otimes \boldsymbol{u} = \lambda \otimes \boldsymbol{u}$, which is the same as the defining equation of usual eigenvalues and eigenvectors. In fact, we will prove later that eigenvectors of A with respect to other eigenvalues are also algebraic eigenvectors.

Proof. "If part": Suppose there exists a λ-maximal multi-circuit $\mathcal{C}$ and a vector $\boldsymbol{u} \neq \mathcal{E}$ such that

$$(A_{\setminus\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{u} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus\mathcal{C}}) \otimes \boldsymbol{u}.$$

For $i = 1, 2, \dots, 2n$, if we evaluate the ith row of

$$\begin{pmatrix} A & \lambda \otimes E \\ E & E \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{u} \end{pmatrix},$$

the maximum is attained at least twice. This means that $\tilde{A}(\lambda)$ is singular by Proposition 4.4. Hence, λ is an algebraic eigenvalue of A by Proposition 7.3.

"Only if" part: Suppose λ is an algebraic eigenvalue of A. First, we consider the case $\lambda \neq \varepsilon$. From Proposition 4.4 and 7.3, there exists a nontrivial vector $\tilde{\boldsymbol{v}} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{v} \end{pmatrix} \in \mathbb{R}_{\max}^{2n}$ such that the maximum of each row of

$$\begin{pmatrix} A & \lambda \otimes E \\ E & E \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{v} \end{pmatrix}$$

is attained at least twice. Let $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. We define the matrices P and Q and the vector $\boldsymbol{b}$ by

$$P = \begin{pmatrix} A_{\mathcal{C}} & \lambda \otimes E_{\setminus\mathcal{C}} \\ E_{\setminus\mathcal{C}} & E_{\mathcal{C}} \end{pmatrix}, \quad Q = \begin{pmatrix} A_{\setminus\mathcal{C}} & \lambda \otimes E_{\mathcal{C}} \\ E_{\mathcal{C}} & E_{\setminus\mathcal{C}} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} (A \oplus \lambda \otimes E) \otimes \boldsymbol{v} \\ \boldsymbol{v} \end{pmatrix}.$$

Since the (i, j) entry of P is finite if and only if $j = \pi^{\mathcal{C}}(i)$, P has its inverse P^{-1}. We consider the equation $P \otimes \tilde{\boldsymbol{x}} = Q \otimes \tilde{\boldsymbol{x}} \oplus \boldsymbol{b}$ and its solution of the form

$$\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{u}} := (P^{-1} \otimes Q)^* \otimes (P^{-1} \otimes \boldsymbol{b}).$$

Then, the vector consisting of the first n entries of $\tilde{\boldsymbol{u}}$ is the desired algebraic eigenvector. Indeed, since we compute

$$\tilde{\boldsymbol{u}} = (P^{-1} \otimes Q)^* \otimes P^{-1} \otimes (P \oplus Q) \otimes \tilde{\boldsymbol{v}} = (P^{-1} \otimes Q)^* \otimes \tilde{\boldsymbol{v}},$$

we have $\tilde{\boldsymbol{u}} \geq \tilde{\boldsymbol{v}}$. From our choice of $\tilde{\boldsymbol{v}}$, we have

$$Q \otimes \tilde{\boldsymbol{u}} \geq Q \otimes \tilde{\boldsymbol{v}} = (P \oplus Q) \otimes \tilde{\boldsymbol{v}} = \boldsymbol{b}.$$

Thus, we obtain $P \otimes \tilde{\boldsymbol{u}} = Q \otimes \tilde{\boldsymbol{u}}$. By the last n rows of this equation, $\tilde{\boldsymbol{u}}$ is of the form $\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{u} \end{pmatrix}$. Checking the first n rows, we have

$$(A_{\setminus \mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{u} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus \mathcal{C}}) \otimes \boldsymbol{u}.$$

Next, we consider the case $\lambda = \varepsilon$. Let $\mathcal{C}$ be an ε-maximal multi-circuit. For sufficiently small number t, we define

$$P_t = \begin{pmatrix} A_{\mathcal{C}} & t \otimes E_{\setminus \mathcal{C}} \\ E_{\setminus \mathcal{C}} & E_{\mathcal{C}} \end{pmatrix}, \quad Q_t = \begin{pmatrix} A_{\setminus \mathcal{C}} & t \otimes E_{\mathcal{C}} \\ E_{\mathcal{C}} & E_{\setminus \mathcal{C}} \end{pmatrix}, \quad \boldsymbol{b}_t = {}^t(t, t, \ldots, t).$$

The vector $\tilde{\boldsymbol{u}}_t = (P_t^{-1} \otimes Q_t)^* \otimes (P_t^{-1} \otimes \boldsymbol{b}_t)$ satisfies $P_t \otimes \tilde{\boldsymbol{u}}_t = Q_t \otimes \tilde{\boldsymbol{u}}_t \oplus \boldsymbol{b}_t$. Taking the limit $t \to -\infty$, we obtain the desired vector as the first n entries of $\tilde{\boldsymbol{u}} := \lim_{t \to -\infty} \tilde{\boldsymbol{u}}_t$. The fact that $\tilde{\boldsymbol{u}} \in \mathbb{R}_{\max}^{2n} \setminus \{\mathcal{E}\}$ can be proved as follows. We first verify that all entries of $\tilde{\boldsymbol{u}}_t$ are of the form $c + dt, d \geq 0$, by easy computations, which implies $\tilde{\boldsymbol{u}} \in \mathbb{R}_{\max}^{2n}$. We next see that $\tilde{\boldsymbol{u}}$ is nontrivial. Since $\lambda = \varepsilon$ is an algebraic eigenvalue, $V(\mathcal{C})$ must not be $\{1, 2, \ldots, n\}$. Take $k \notin V(\mathcal{C})$. From the kth and $(k+n)$th rows of $P_t \otimes \tilde{\boldsymbol{u}}_t = Q_t \otimes \tilde{\boldsymbol{u}}_t \oplus \boldsymbol{b}_t$, we have

$$\begin{aligned} &[P_t]_{k\,k+n} \otimes [\tilde{\boldsymbol{u}}_t]_{k+n} \geq [\boldsymbol{b}_t]_k, \\ &[P_t]_{k+n\,k} \otimes [\tilde{\boldsymbol{u}}_t]_k \geq [Q_t]_{k+n\,k+n} \otimes [\tilde{\boldsymbol{u}}_t]_{k+n}, \end{aligned}$$

Since $[P_t]_{k\,k+n} = [\boldsymbol{b}_t]_k = t$ and $[P_t]_{k+n\,k} = [Q_t]_{k+n\,k+n} = 0$, we have

$$[\tilde{\boldsymbol{u}}_t]_k \geq [\tilde{\boldsymbol{u}}_t]_{k+n} \geq 0.$$

As this holds for arbitrary small value t, $[\tilde{\boldsymbol{u}}_t]_k$ is a finite constant independent of t. Thus, $[\tilde{\boldsymbol{u}}]_k \neq \varepsilon$. □

Example 7.6. Let us consider the max-plus matrix

$$A = \begin{pmatrix} \varepsilon & 9 & 8 & \varepsilon & 0 & \varepsilon \\ 7 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon \\ 6 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

The characteristic polynomial of A is

$$\varphi_A(t) = (t \oplus 8)^{\otimes 2} \otimes (t \oplus 2)^{\otimes 2} \otimes t^{\otimes 2}.$$

Take an algebraic eigenvalue 2 of A and a 2-maximal multi-circuit $\mathcal{C} = \{(1, 2, 1)\}$. Then, the defining equation of algebraic eigenvectors in Theorem 7.4 is

$$\left(\begin{pmatrix} \varepsilon & \varepsilon & 8 & \varepsilon & 0 & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon \\ 6 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \oplus 2 \otimes \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \right) \otimes \boldsymbol{u}$$

$$= \left(\begin{pmatrix} \varepsilon & 9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 7 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \oplus 2 \otimes \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \right) \otimes \boldsymbol{u}.$$

It can be easily verified that $\boldsymbol{u} = {}^t(0, 5, 6, 4, 4, 5)$ is an algebraic eigenvector of A with respect to algebraic eigenvalue 2. We will show later in Example 7.10 how to find this algebraic eigenvector.

7.3 Algebraic eigenspaces

Next, we describe the set of all algebraic eigenvectors. Let $A \in \mathbb{R}_{\max}^{n \times n}$. For an algebraic eigenvalue λ of A and a multi-circuit $\mathcal{C}$ in $G(A)$, we define

$$W(A, \lambda, \mathcal{C}) = \left\{ \boldsymbol{u} \in \mathbb{R}_{\max}^n \;\middle|\; (A_{\setminus \mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{u} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus \mathcal{C}}) \otimes \boldsymbol{u} \right\}.$$

Lemma 7.7. *Let $\lambda \neq \varepsilon$ be an algebraic eigenvalue of $A \in \mathbb{R}_{\max}^{n \times n}$ and $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. Then, for all multi-circuits $\mathcal{C}'$ in $G(A)$, we have $W(A, \lambda, \mathcal{C}') \subset W(A, \lambda, \mathcal{C})$.*

Proof. Let $\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_n) \in W(A, \lambda, \mathcal{C}')$. We set $u_{j+n} = u_j$ for $j = 1, 2, \ldots, n$. Then, we have

$$\tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes u_{\pi^{\mathcal{C}}(i)} \leq \bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes u_j = \tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes u_{\pi^{\mathcal{C}'}(i)}$$

for $i = 1, 2, \ldots, 2n$, where $\tilde{A}(\lambda) = (\tilde{a}_{ij})$. We first assume all entries of $\boldsymbol{u}$ are finite. Then, we have

$$\begin{aligned}
\bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes u_{\pi^{\mathcal{C}}(i)} \leq \bigotimes_{i=1}^{2n} \bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes u_j &= \bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes u_{\pi^{\mathcal{C}'}(i)} \\
&= \bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes \bigotimes_{i=1}^{2n} u_{\pi^{\mathcal{C}'}(i)} \\
&\leq \bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes \bigotimes_{i=1}^{2n} u_{\pi^{\mathcal{C}}(i)} \\
&= \bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes u_{\pi^{\mathcal{C}}(i)},
\end{aligned}$$

proving that

$$\bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes u_{\pi^{\mathcal{C}}(i)} = \bigotimes_{i=1}^{2n} \bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes u_j.$$

As this value is finite, we have

$$\tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes u_{\pi^{\mathcal{C}}(i)} = \bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes u_j, \quad i = 1, 2, \ldots, 2n.$$

In particular, we have proved that $\boldsymbol{u} \in W(A, \lambda, \mathcal{C})$ from the equalities for $i = 1, 2, \ldots, n$.

Next, we assume that some but not all entries of $\boldsymbol{u} \in W(A, \lambda, \mathcal{C}')$ are ε. Let $K = \{j \mid u_j \neq \varepsilon\}$ and $L = \{j \mid u_j = \varepsilon\}$. For $i \in K$, since we have

$$\tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes u_{\pi^{\mathcal{C}'}(i)} = \bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes u_j \geq \lambda \otimes u_{i+n} = \lambda \otimes u_i \neq \varepsilon,$$

we obtain $u_{\pi^{\mathcal{C}'}(i)} \neq \varepsilon$. This implies $\pi^{\mathcal{C}'}(K) = K$ and hence $\pi^{\mathcal{C}'}(L) = L$. For $i \in L$ and $k \in K$, we have

$$\tilde{a}_{ik} \otimes u_k \leq \bigotimes_{j=1}^{2n} \tilde{a}_{ij} \otimes u_j = \tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes u_{\pi^{\mathcal{C}'}(i)} = \varepsilon.$$

Thus $\tilde{a}_{ik}$ must be ε. Since $\det \tilde{A}(\lambda) \neq \varepsilon$ for any finite value λ, $\pi^{\mathcal{C}}$ satisfies $\pi^{\mathcal{C}}(K) = K$ and $\pi^{\mathcal{C}}(L) = L$. Restricting calculations only to the rows and columns indexed by K, we obtain $\boldsymbol{u} \in W(A, \lambda, \mathcal{C})$ by the same argument as above. $\square$

Let λ be a finite algebraic eigenvalue of A. For two distinct λ-maximal multi-circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ in $G(A)$, we have both $W(A, \lambda, \mathcal{C}_1) \subset W(A, \lambda, \mathcal{C}_2)$ and $W(A, \lambda, \mathcal{C}_2) \subset W(A, \lambda, \mathcal{C}_1)$, which implies that the set $W(A, \lambda, \mathcal{C})$ does not depend on the choice of λ-maximal multi-circuit $\mathcal{C}$. On the other hand, if $\lambda = \varepsilon$, the λ-maximal multi-circuit is unique under Assumption 7.2. Thus, we write $W(A, \lambda) := W(A, \lambda, \mathcal{C})$, where $W(A, \lambda)$ is the set of all algebraic eigenvectors of A with respect to λ. Since $W(A, \lambda)$ is the set of solutions of a homogeneous linear system, $W(A, \lambda)$ is a max-plus subspace of $\mathbb{R}^n_{\max}$. Hence, it is called the *algebraic eigenspace* of A with respect to λ. We also see that any usual eigenspace $U(A, \lambda)$ is contained in the algebraic eigenspace $W(A, \lambda)$ by setting $\mathcal{C}' = \emptyset$ in Lemma 7.7.

7.4 Dimensions and multiplicities

In this subsection, we give an upper bound for the dimension of the algebraic eigenspace of a matrix by the multiplicity of the algebraic eigenvalue.

Theorem 7.8 ([64])**.** *Let λ be an algebraic eigenvalue of $A \in \mathbb{R}^{n \times n}_{\max}$. Then, the dimension of the algebraic eigenspace $W(A, \lambda)$ does not exceed the multiplicity of the root λ in the characteristic polynomial $\varphi_A(t)$.*

To prove this theorem, we distinguish the case where λ is finite from the case $\lambda = \varepsilon$. We first consider the case $\lambda \neq \varepsilon$. Let $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. Then, an algebraic eigenvector $\boldsymbol{u} \in W(A, \lambda)$ satisfies

$$(A_{\setminus \mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{u} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus \mathcal{C}}) \otimes \boldsymbol{u}.$$

Since $A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus \mathcal{C}}$ is invertible, we have

$$\big((A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus \mathcal{C}})^{-1} \otimes (A_{\setminus \mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}})\big) \otimes \boldsymbol{u} = \boldsymbol{u}.$$

This means that $\boldsymbol{u}$ is an eigenvector of $B_{\mathcal{C}} := (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus \mathcal{C}})^{-1} \otimes (A_{\setminus \mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}})$ with respect to the eigenvalue 0 of $B_{\mathcal{C}}$. Thus, from Theorem 5.8, we see that the dimension of the algebraic eigenspace $W(A, \lambda)$ is the number of connected components of $G_c(B_{\mathcal{C}})$.

Lemma 7.9. *Let $\lambda \neq \varepsilon$ be an algebraic eigenvalue of $A \in \mathbb{R}^{n \times n}_{\max}$ and $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. Then, from any multi-circuit $\mathcal{D}$ with weight 0 in $G(B_{\mathcal{C}})$, we can find a multi-circuit $\mathcal{C}'$ in $G(A)$ satisfying*

$$(\ell(\mathcal{C}') - \ell(\mathcal{C}))\lambda = w(\mathcal{C}') - w(\mathcal{C})$$

and

$$(V(\mathcal{C}') \setminus V(\mathcal{C})) \cup (V(\mathcal{C}) \setminus V(\mathcal{C}')) \subset V(\mathcal{D}).$$

It follows from the first equality that $\mathcal{C}'$ is also a λ-maximal multi-circuit in $G(A)$.

Since the proof of this lemma is quite complicated and rather technical, we leave it in Section 7.5 later.

Proof of Theorem 7.8 for the case $\lambda \neq \varepsilon$. Let m be the dimension of the algebraic eigenspace $W(A,\lambda)$ and $\mathcal{C}$ be a λ-maximal multi-circuit with the minimum length in $G(A)$. Then, there are m disjoint circuits $D_1, D_2, \ldots, D_m$ with (average) weights 0 in $G(B_{\mathcal{C}})$. It follows from Lemma 7.9 that we find λ-maximal circuits $\mathcal{C}_i, i = 1,2,\ldots,m$, corresponding to D_i. Let $\mathcal{C}'$ be the λ-maximal multi-circuit obtained from $\mathcal{D} := \{D_1, D_2, \ldots, D_m\}$. Since we have

$$\begin{aligned}
&\big((V(\mathcal{C}_i) \setminus V(\mathcal{C})) \cup (V(\mathcal{C}) \setminus V(\mathcal{C}_i))\big) \cap \big((V(\mathcal{C}_j) \setminus V(\mathcal{C})) \cup (V(\mathcal{C}) \setminus V(\mathcal{C}_j))\big)\\
\subset& V(D_i) \cap V(D_j)\\
=&\emptyset
\end{aligned}$$

if $i \neq j$, we see that the construction of the λ-maximal multi-circuit $\mathcal{C}_i$ from each circuit D_i in $B_{\mathcal{C}}$ does not interfere with each other. Hence, we have

$$V(\mathcal{C}') \setminus V(\mathcal{C}) = \bigcup_{i=1}^{m} (V(\mathcal{C}_i) \setminus V(\mathcal{C})), \quad V(\mathcal{C}) \setminus V(\mathcal{C}') = \bigcup_{i=1}^{m} (V(\mathcal{C}) \setminus V(\mathcal{C}_i)).$$

From Assumption 7.2 and the minimality of the length of $\mathcal{C}$, we see $\ell(\mathcal{C}_i) - \ell(\mathcal{C}) \geq 1$ for all $i = 1,2,\ldots,m$. Further we note

$$\ell(\mathcal{C}_i) - \ell(\mathcal{C}) = |V(\mathcal{C}_i)| - |V(\mathcal{C})| = |V(\mathcal{C}_i) \setminus V(\mathcal{C})| - |V(\mathcal{C}) \setminus V(\mathcal{C}_i)|$$

for $i = 1,2,\ldots,m$. Thus, we have

$$\begin{aligned}
\ell(\mathcal{C}') - \ell(\mathcal{C}) &= |V(\mathcal{C}') \setminus V(\mathcal{C})| - |V(\mathcal{C}) \setminus V(\mathcal{C}')|\\
&= \sum_{i=1}^{m} \left(|V(\mathcal{C}_i) \setminus V(\mathcal{C})| - |V(\mathcal{C}) \setminus V(\mathcal{C}_i)|\right)\\
&\geq m.
\end{aligned}$$

This means there exists a multi-circuit $\mathcal{C}'$ in $G(A)$ satisfying $\ell(\mathcal{C}') \geq \ell(\mathcal{C}) + m$ and $\mathrm{r.ave}(\mathcal{C}, \mathcal{C}') = \lambda$. Algorithm 5.15 implies that m cannot exceed the multiplicity of λ. $\square$

To prove the case $\lambda = \varepsilon$, we use the iterative method in Section 4.

Proof of Theorem 7.8 for the case $\lambda = \varepsilon$. Suppose $\mathcal{C}$ is the ε-maximal multi-circuit of length ℓ in $G(A)$. We assume without loss of generality that $V(\mathcal{C}) = \{1,2,\ldots,\ell\}$. Let $\boldsymbol{u} \in W(A,\varepsilon)$ and $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ be the first ℓ rows and the last $(n-\ell)$ rows of $\boldsymbol{u}$. Then we have

$$\begin{pmatrix} A^1_{\setminus\mathcal{C}} & A^2 \\ A^3 & A^4 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{u}^1 \\ \boldsymbol{u}^2 \end{pmatrix} = \begin{pmatrix} A^1_{\mathcal{C}} & \mathcal{E} \\ \mathcal{E} & \mathcal{E} \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{u}^1 \\ \boldsymbol{u}^2 \end{pmatrix},$$

where $A = \begin{pmatrix} A^1 & A^2 \\ A^3 & A^4 \end{pmatrix}$, $A^1 \in \mathbb{R}_{\max}^{\ell\times\ell}, A^2 \in \mathbb{R}_{\max}^{\ell\times(n-\ell)}, A^3 \in \mathbb{R}_{\max}^{(n-\ell)\times\ell}, A^4 \in \mathbb{R}_{\max}^{(n-\ell)\times(n-\ell)}$, yielding two equations:

$$\begin{aligned} A^1_{\backslash\mathcal{C}} \otimes \boldsymbol{u}^1 \oplus A^2 \otimes \boldsymbol{u}^2 &= A^1_{\mathcal{C}} \otimes \boldsymbol{u}^1, \\ A^3 \otimes \boldsymbol{u}^1 \oplus A^4 \otimes \boldsymbol{u}^2 &= \mathcal{E}. \end{aligned}$$

Fix a vector $\boldsymbol{u}^2 \in \mathbb{R}_{\max}^{n-\ell}$. By Corollary 4.2, the first equation has the unique solution

$$\boldsymbol{u}^1 = \xi(\boldsymbol{u}^2) := \left((A^1_{\mathcal{C}})^{-1} \otimes A^1_{\backslash\mathcal{C}}\right)^* \otimes \left((A^1_{\mathcal{C}})^{-1} \otimes A^2 \otimes \boldsymbol{u}^2\right)$$

because every circuit in $G((A^1_{\mathcal{C}})^{-1} \otimes A^1_{\backslash\mathcal{C}})$ has negative weight by Assumption 7.2. Combining this solution with the second equation, we have

$$W(A,\varepsilon) = \left\{ \boldsymbol{u} = \begin{pmatrix} \xi(\boldsymbol{u}^2) \\ \boldsymbol{u}^2 \end{pmatrix} \;\middle|\; [\boldsymbol{u}]_j = \varepsilon \text{ if } a_{ij} \neq \varepsilon \text{ for some } i = \ell+1, \ldots, n \right\}.$$

In particular, a basis of $W(A,\varepsilon)$ is the set

$$\left\{ \begin{pmatrix} \xi(\tilde{\boldsymbol{e}}_k) \\ \tilde{\boldsymbol{e}}_k \end{pmatrix} \;\middle|\; \begin{array}{l} \text{the } k\text{th column of } A^4 \text{ is } \mathcal{E}, \\ [\xi(\tilde{\boldsymbol{e}}_k)]_j = \varepsilon \text{ if } a_{ij} \neq \varepsilon \text{ for some } i = \ell+1, \ldots, n \end{array} \right\},$$

where $\tilde{\boldsymbol{e}}_k, 1 \le k \le n-\ell$, are the standard basis vectors of $\mathbb{R}_{\max}^{n-\ell}$. Hence, the dimension of $W(A,\varepsilon)$ does not exceed $n-\ell$, which is the multiplicity of the root ε in $\varphi_A(t)$. □

Example 7.10. We again consider

$$A = \begin{pmatrix} \varepsilon & 9 & 8 & \varepsilon & 0 & \varepsilon \\ 7 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon \\ 6 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

The algebraic eigenspace $W(A,8)$ is the same as the eigenspace of A with respect to 8. Hence, computing

$$((-8) \otimes A)^* = \begin{pmatrix} 0 & 1 & 0 & -4 & -8 & -15 \\ -1 & 0 & -1 & -5 & -9 & -16 \\ -6 & -5 & 0 & -4 & -8 & -15 \\ -2 & -1 & -2 & 0 & -10 & -17 \\ -14 & -13 & -14 & -18 & 0 & -7 \\ -7 & -6 & -7 & -11 & -15 & 0 \end{pmatrix},$$

we identify a basis ${}^t(0, -1, -6, -2, -14, -7)$ of $W(8)$.

From the discussion after the statement of Theorem 7.8, the algebraic eigenspace $W(A,2)$ is same as the eigenspace of

$$B_{\{(1,2,1)\}} = (A_{\{(1,2,1)\}} \oplus 2 \otimes E_{\setminus\{(1,2,1)\}})^{-1} \otimes (A_{\setminus\{(1,2,1)\}} \oplus 2 \otimes E_{\{(1,2,1)\}})$$
$$= \begin{pmatrix} \varepsilon & -5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ -7 & \varepsilon & -1 & \varepsilon & -9 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon \\ 4 & -2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -1 \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

We see $G(B_{\{(1,2,1)\}})$ has exactly one circuit $(1,2,3,4,1)$ with average weight 0. Computing $(B_{\{(1,2,1)\}})^*$, we have a basis ${}^t(0,5,6,4,4,5)$ of $W(A,2)$.

For the algebraic eigenvalue ε, the ε-maximal multi-circuit in $G(A)$ is $\mathcal{C} = \{(1,3,4,1),(2,2)\}$. The map $\xi : \mathbb{R}^2_{\max} \to \mathbb{R}^4_{\max}$ in the above proof is given by

$$\xi(\boldsymbol{u}^2) = \left(\begin{pmatrix} \varepsilon & \varepsilon & 8 & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 \\ 6 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}^{-1} \otimes \begin{pmatrix} \varepsilon & 9 & \varepsilon & \varepsilon \\ 7 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix} \right)^*$$
$$\otimes \left(\begin{pmatrix} \varepsilon & \varepsilon & 8 & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 \\ 6 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}^{-1} \otimes \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \right) \otimes \boldsymbol{u}^2.$$

Since the first column of the right bottom 2×2 block of A is ${}^t(\varepsilon,\varepsilon)$, the vector $({}^t\xi(\tilde{\boldsymbol{e}}_1), {}^t\tilde{\boldsymbol{e}}_1) = {}^t(\varepsilon,\varepsilon,-8,\varepsilon,0,\varepsilon)$ is a basis of $W(A,\varepsilon)$.

Thus, we have computed bases of all algebraic eigenspaces of A and have found that the dimensions of all algebraic eigenspaces are 1, which is less than their multiplicities in $\varphi_A(t) = (t \oplus 8)^{\otimes 2} \otimes (t \oplus 2)^{\otimes 2} \otimes t^{\otimes 2}$.

7.5 Proof of Lemma 7.9

Let $\lambda \neq \varepsilon$ be an algebraic eigenvalue of $A \in \mathbb{R}^{n \times n}_{\min}$ and $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. For any vertex $i \in V(\mathcal{C})$, we denote by $\sigma(i)$ the succeeding vertex of i in the circuit in $\mathcal{C}$; $\sigma^{-1}(i)$ is the preceding vertex of i in $\mathcal{C}$. Recalling that $B_{\mathcal{C}} := (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus\mathcal{C}})^{-1} \otimes (A_{\setminus\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}})$ and computing entries of $B_{\mathcal{C}}$, we obtain the correspondence in edges between $G(A)$ and $G(B_{\mathcal{C}})$ shown in Table 1. We see that $G(B_{\mathcal{C}})$ has a multi-circuit $\overleftarrow{\mathcal{C}}$, which consists of the edges $\{(\sigma(i), i) \mid i \in V(\mathcal{C})\}$.

Let $\mathcal{D}$ be a multi-circuit in $G(B_{\mathcal{C}})$ with (average) weight 0. We construct a multi-circuit $\mathcal{C}'$ in $G(A)$ by the following steps:

Table 1: Correspondence between $G(A)$ and $G(B_{\mathcal{C}})$

$G(A)$		$G(B_{\mathcal{C}})$	
edge	weight	edge	weight
$(i,i'), i \notin V(\mathcal{C})$	$a_{ii'}$	(i,i')	$-\lambda + a_{ii'}$
$(i,\sigma(i)), i \in V(\mathcal{C})$	$a_{i\sigma(i)}$	$(\sigma(i), i)$	$\max\{\lambda, a_{ii}\} - a_{i\sigma(i)}$
$(i,i'), i \in V(\mathcal{C}), i' \neq \sigma(i)$	$a_{ii'}$	$(\sigma(i), i')$	$a_{ii'} - a_{i\sigma(i)}$

(1) Set $\mathcal{C}' := \emptyset$.

(2) Choose any edge of $\mathcal{D}$ that is not in $E(\overleftarrow{\mathcal{C}})$ and denote the terminal vertex of that edge by i. We define the initial sequence of vertices by $\hat{C} := (i)$.

(3) The succeeding vertex of i in $\hat{C}$ is determined by the following rules.

   (a) If $i \notin V(\mathcal{C})$, let i' be the succeeding vertex of i in $\mathcal{D}$. Append i' to $\hat{C}$ and set $i := i'$.

   (b) If $i \in V(\mathcal{C})$ and $\sigma(i) \notin V(\mathcal{D})$, append $\sigma(i)$ to $\hat{C}$ and set $i := \sigma(i)$.

   (c) If $i \in V(\mathcal{C})$ and $\sigma(i) \in V(\mathcal{D})$, let i' be the succeeding vertex of $\sigma(i)$ in $\mathcal{D}$. Append i' to $\hat{C}$ and set $i := i'$.

(4) Repeat (3) until the original vertex i selected in (2) appears again. If we return to i, append the circuit $\hat{C}$ to $\mathcal{C}'$.

(5) Repeat (2)–(4) while there exist edges (or corresponding terminal vertices) satisfying (2).

(6) Append all circuits in $\mathcal{C}$ that have no common vertices with $\mathcal{D}$ to $\mathcal{C}'$.

(7) Find all loops on $V(\mathcal{D}) \setminus V(\mathcal{C}')$ whose weights are greater than λ. Append them to $\mathcal{C}'$.

An example of these steps is illustrated in Figure 7. The steps (2)–(5) give a union of disjoint circuits because this vertex search is uniquely traced back as follows:

- If $j \in V(\mathcal{D})$, let j' be the preceding vertex of j in $\mathcal{D}$. The preceding vertex of j in $\hat{C}$ is $\sigma^{-1}(j')$ if $j' \in \mathcal{C}$; otherwise it is j'.
- If $j \notin V(\mathcal{D})$, the preceding vertex of j in $\hat{C}$ is $\sigma^{-1}(j)$.

Lemma 7.11. *Let $\mathcal{C}'$ be the multi-circuit in $G(A)$ constructed as above. We have $V(\mathcal{C}) \setminus V(\mathcal{C}') \subset V(\mathcal{D})$ and $(\sigma(i), i) \in E(\mathcal{D})$ for any $i \in V(\mathcal{C}) \setminus V(\mathcal{C}')$.*

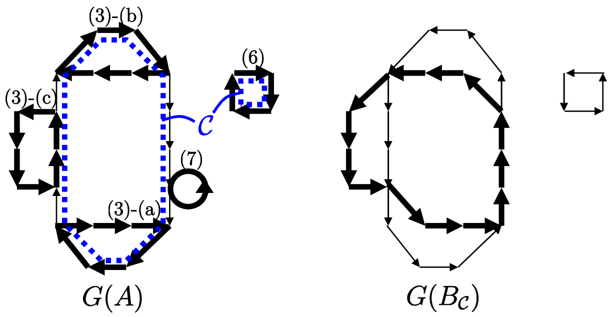

Figure 7: Graph $G(A)$ and circuit $\mathcal{C}$ (left) and graph $G(B_{\mathcal{C}})$ (right). Bold arrows represent the multi-circuit $\mathcal{D}$ (right) and the corresponding multi-circuit $\mathcal{C}'$ (left). Examples of five types of edges are also illustrated.

Proof. We assume the contrary. Suppose there is a vertex $j \in (V(\mathcal{C}) \setminus V(\mathcal{C}')) \setminus V(\mathcal{D})$. In that case, we show that, without loss of generality, we may assume $\sigma^{-1}(j) \in V(\mathcal{D})$. In order to show the assumption is proper, first we prove $\sigma^{-1}(j) \notin V(\mathcal{C}')$ if $\sigma^{-1}(j) \notin V(\mathcal{D})$: if $\sigma^{-1}(j) \in V(\mathcal{C}') \setminus V(\mathcal{D})$, then step (3) should has been executed for $i := \sigma^{-1}(j)$ just after the vertex $\sigma^{-1}(j)$ was appended to $V(\mathcal{C}')$. Since $\sigma^{-1}(j) \in V(\mathcal{C})$ and $j = \sigma(\sigma^{-1}(j)) \notin V(\mathcal{D})$, the case (b) occurs and we have $j \in V(\mathcal{C}')$, leading to a contradiction. Thus, we can continue to replace j with $\sigma^{-1}(j)$ until $\sigma^{-1}(j)$ is contained in $V(\mathcal{D})$. The edge in $\mathcal{D}$ whose terminal vertex is $\sigma^{-1}(j)$ exists but it is not $(j, \sigma^{-1}(j))$ since $j \notin V(\mathcal{D})$. Thus, $\sigma^{-1}(j)$ must be in $V(\mathcal{C}')$ and the case (3)-(b) occurs for $i := \sigma^{-1}(j)$, which implies $j \in V(\mathcal{C}')$, leading to a contradiction. Hence, we conclude $V(\mathcal{C}) \setminus V(\mathcal{C}') \subset V(\mathcal{D})$, which is the first assertion of the lemma. In particular, if $(\sigma(i), i)$ were not an edge of $\mathcal{D}$ for some $i \in V(\mathcal{C}) \setminus V(\mathcal{C}')$, there would be another edge in $\mathcal{D}$ whose terminal vertex is i. By step (2), this means $i \in V(\mathcal{C}')$, leading to a contradiction. $\square$

Proof of Lemma 7.9. The inclusion $V(\mathcal{C}) \setminus V(\mathcal{C}') \subset V(\mathcal{D})$ has been proved in Lemma 7.11. On the other hand, from the above procedure, each vertex in $V(\mathcal{C}')$ is contained in $V(\mathcal{C})$ or $V(\mathcal{D})$, which shows $V(\mathcal{C}') \setminus V(\mathcal{C}) \subset V(\mathcal{D})$.

We next prove the equality for weights. Let $\{(\alpha_k, \alpha'_k)\}$ be the set of the edges in $\mathcal{C}'$ constructed by (3)-(a), $\{(\beta_k, \sigma(\beta_k))\}$ by (3)-(b), $\{(\gamma_k, \gamma'_k)\}$ by (3)-(c), $\{(\delta_k, \sigma(\delta_k))\}$ by (6), $\{(\epsilon_k, \epsilon_k)\}$ by (7), respectively. We denote by

$\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_\delta$ and ℓ_ϵ the numbers of those edges, respectively. Then, we have

$$\begin{aligned}
&w(\mathcal{C}') - w(\mathcal{C}) \\
=&\sum_{k=1}^{\ell_\alpha} a_{\alpha_k \alpha'_k} + \sum_{k=1}^{\ell_\beta} a_{\beta_k \sigma(\beta_k)} + \sum_{k=1}^{\ell_\gamma} a_{\gamma_k \gamma'_k} + \sum_{k=1}^{\ell_\delta} a_{\delta_k \sigma(\delta_k)} \\
&\qquad\qquad\qquad\qquad + \sum_{k=1}^{\ell_\epsilon} a_{\epsilon_k \epsilon_k} - \sum_{i \in V(\mathcal{C})} a_{i\sigma(i)} \\
=&\sum_{k=1}^{\ell_\alpha} a_{\alpha_k \alpha'_k} + \sum_{k=1}^{\ell_\gamma} a_{\gamma_k \gamma'_k} + \sum_{k=1}^{\ell_\epsilon} a_{\epsilon_k \epsilon_k} - \sum_{i \in V(\mathcal{C}) \setminus \{\beta_k, \delta_k\}} a_{i\sigma(i)}.
\end{aligned}$$

Let b_{ij} be the weight of the edge (i, j) in $G(B_\mathcal{C})$. From Table 1, we have

$$\begin{aligned}
\sum_{k=1}^{\ell_\alpha} a_{\alpha_k \alpha'_k} &= \ell_\alpha \lambda + \sum_{k=1}^{\ell_\alpha} b_{\alpha_k \alpha'_k}, \\
\sum_{k=1}^{\ell_\gamma} a_{\gamma_k \gamma'_k} &= \sum_{k=1}^{\ell_\gamma} a_{\gamma_k \sigma(\gamma_k)} + \sum_{k=1}^{\ell_\gamma} b_{\sigma(\gamma_k)\gamma'_k}, \\
\sum_{k=1}^{\ell_\epsilon} a_{\epsilon_k \epsilon_k} &= \sum_{k=1}^{\ell_\epsilon} (a_{\epsilon_k \sigma(\epsilon_k)} + b_{\sigma(\epsilon_k)\epsilon_k}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&w(\mathcal{C}') - w(\mathcal{C}) \\
=&\ell_\alpha \lambda + \sum_{k=1}^{\ell_\alpha} b_{\alpha_k \alpha'_k} + \sum_{k=1}^{\ell_\gamma} b_{\sigma(\gamma^k)\gamma'^k} + \sum_{k=1}^{\ell_\epsilon} b_{\sigma(\epsilon_k)\epsilon_k} - \sum_{i \in V(\mathcal{C}) \setminus \{\beta_k, \gamma_k, \delta_k, \epsilon_k\}} a_{i\sigma(i)} \\
=&\ell_\alpha \lambda + \sum_{k=1}^{\ell_\alpha} b_{\alpha_k \alpha'_k} + \sum_{k=1}^{\ell_\gamma} b_{\sigma(\gamma^k)\gamma'^k} + \sum_{k=1}^{\ell_\epsilon} b_{\sigma(\epsilon_k)\epsilon_k} \\
&\qquad - \left((\ell(\mathcal{C}) - \ell_\beta - \ell_\gamma - \ell_\delta - \ell_\epsilon)\lambda - \sum_{i \in V(\mathcal{C}) \setminus \{\beta_k, \gamma_k, \delta_k, \epsilon_k\}} b_{\sigma(i)i} \right) \\
=&(\ell(\mathcal{C}') - \ell(\mathcal{C}))\lambda + \sum_{k=1}^{\ell_\alpha} b_{\alpha_k \alpha'_k} + \sum_{k=1}^{\ell_\gamma} b_{\sigma(\gamma^k)\gamma'^k} \\
&\qquad\qquad\qquad + \sum_{k=1}^{\ell_\epsilon} b_{\sigma(\epsilon_k)\epsilon_k} + \sum_{i \in V(\mathcal{C}) \setminus \{\beta_k, \gamma_k, \delta_k, \epsilon_k\}} b_{\sigma(i)i}.
\end{aligned}$$

From our procedure and Lemma 7.11, we have

$$\begin{aligned} E(\mathcal{D}) = &\{(\alpha_k, \alpha'_k) \mid k = 1, 2, \ldots, \ell_\alpha\} \\ &\cup \{(\sigma(\gamma_k), \gamma'_k) \mid k = 1, 2, \ldots, \ell_\gamma\} \\ &\cup \{(\sigma(\epsilon_k), \epsilon_k) \mid k = 1, 2, \ldots, \ell_\epsilon\} \\ &\cup \{(\sigma(i), i) \mid i \in V(\mathcal{C}) \setminus \{\beta_k, \gamma_k, \delta_k, \epsilon_k\}\}. \end{aligned}$$

Thus, we obtain

$$w(\mathcal{C}') - w(\mathcal{C}) = (\ell(\mathcal{C}') - \ell(\mathcal{C}))\lambda + w(\mathcal{D}).$$

Since $w(\mathcal{D}) = 0$, we have the desired equality. $\square$

8 Conclusion

In the present thesis, the author develops the linear algebra on the max-plus algebra. The theory on linear systems and the eigenvalue problem are two major themes on the study of the max-plus linear algebra. Solving tropical linear systems is so difficult that no polynomial-time algorithm has introduced so far. First, we have presented a characterization of the vectors in a basis of the kernel of a matrix, which is the solution set of a homogeneous tropical linear system. Each vector is a Cramer vector, that is, the vector obtained by applying the Cramer's rule to some submatrix. This characterization provides some information that the basis should have and helps us with theoretical analysis of the kernel. This result also suggests the way to compute the basis of the tropical kernel a matrix. However, the number of Cramer vectors increases exponentially depending on the size of the matrix. Hence, it remains as a future work to derive an efficient method to detect Cramer vectors that are indispensable for the basis.

In the latter part, the author discusses the eigenvalue problem on the max-plus algebra. Since a general max-plus matrix has a few numbers of eigenvalues and eigenvectors, it seems impossible to be diagonalized. Hence, the author has proposed Jordan canonical forms of max-plus matrices. Although they are not based on the SN (or the Jordan-Chevalley) decomposition, a transformation matrix leading to a Jordan canonical form consists of a basis of the generalized eigenspace. Unlike in the conventional linear algebra, not all square matrices have their Jordan canonical forms. Indeed, the author has proved that a matrix can be transformed into a Jordan canonical form if and only if critical circuits of the associated graph cover all vertices and are mutually disjoint. This condition is so restricted that many classes of matrices do not satisfy it.

On the other hand, the concept of algebraic eigenvectors is defined for a more general class of matrices. It is based on the perspective that many roots of the characteristic polynomial of a matrix are not its eigenvalues and seem useful for extending the definition of eigenvalues. The author has derived the fact that the dimension of the algebraic eigenspace with respect to a root of the characteristic polynomial does not exceed the multiplicity of the root. Thus, it is expected that algebraic eigenvectors become tools to establish a block diagonalization theory for general matrices, which is left as a final goal of the max-plus eigenvalue problem.

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