Capital Expansion and Reduction with Fixed and Proportional Costs under Demand Ambiguity*

Motoh Tsujimura†

1 Introduction
2 Firm's Problem
3 Quasi-variational Inequalities for the Firm's Problem
4 Viscosity Solutions of the Firm's Problem
5 Solution of the Quasi-variational Inequalities of the Firm's Problem
6 Numerical Analysis
7 Conclusion

Abstract

This study investigates the capital expansion and reduction problem in firms when the output demand is ambiguous. We use the concept of κ-ignorance to describe the output demand ambiguity. When a firm changes its level of capital, it incurs both fixed and proportional costs. Therefore, the firm's problem is formulated as a stochastic impulse control problem. We solve the firm's problem via quasi-variational inequalities and numerically derive an optimal capital expansion and reduction policy. Further, we numerically conduct a comparative statics analysis that confirms that output demand ambiguity promotes capital reduction, while delaying capital expansion. Additionally, combining both the effects shrinks the continuation region.

Keywords : Capital expansion and reduction ; quasi-variational inequalities ; stochastic impulse control ; ambiguity ; κ-ignorance

1 Introduction

This study investigates how uncertainty affects a firm's decision-making. Specifically, we focus on the effect of uncertainty on the firm's capital investment problem as capital investment plays an important role in business activities. We employ the concept of ambiguity to control deep uncertainty. Ellsberg (1961) shows that agents are often unable to achieve a unique probability distribution in a reference state space. Such uncertain environments are

---

*This research was partially supported by JSPS KAKENHI Grant Number JP18K01714 and Doshisha University All Doshisha Research Model "Urgent Research Related to COVID-19."

†Corresponding author
Address : Karasuma-Higashi-iru, Imadegawa-dori, Kamigyo-ku, Kyoto 602-8580, JAPAN
Phone : +81-75-251-4582 ; E-mail : mtsujimu@mail.doshisha.ac.jp

referred to as ambiguity or Knightian uncertainty (Knight, 1921). We assume that a firm's manager is ambiguity-averse. Gilboa (1987) and Schmeidler (1989) provide an axiomatic setting of ambiguity aversion, which is characterized by expected utility with non-additive probabilities. Gilboa and Schmeidler (1989) extend their approach by using maxmin expected utility over a set of distributions. In this approach, an agent, who is ambiguity averse, chooses the most unfavorable distribution to maximize the minimum expected utility of that distribution. Chen and Espstein (2002) introduce κ-ignorance and extend the work of Gilboa and Schmeidler (1989) from a static to continuous time setting. In the κ-ignorance approach, the drift in the state of the system is within a range. Specifically, the density generator moves in a range $[-\kappa, \kappa]$, where $\kappa > 0$ describes the degree of ambiguity. Cheng (2013) apply κ-ignorance to an infinite time horizon to solve the optimal stopping problem.

The κ-ignorance approach had been applied to examine irreversible investment problems under ambiguity in past studies (Nishimura and Ozaki, 2007 ; Wang, 2010 ; Trojanowska and Kort, 2010 ; Schröder, 2011 ; Hellmann and Thijssen, 2018 ; Sarkar, 2021). These studies assume that the firms have a single option in capital investment, and formulate the firms' problems as optimal stopping problems. In contrast, we consider that the firms have two options in capital investment, following Tsujimura (2020) who analyzed a firm's problem under output demand risk. One option is to expand the capital when the output demand increases to a sufficient level. The other option is to reduce the capital when the output demand decreases to a sufficient level. Once either option is exercised, the firm incurs fixed and proportional costs. The fixed cost represents costs associated with changing the level of capital, such as the research cost. Meanwhile, the proportional cost represents costs of purchase or sale of capital. The firms can exercise these options as frequently as needed. Therefore, we formulate the firm's problem as a stochastic impulse control problem.

We propose that the solution of quasi-variational inequalities (QVI) is the value function of the firm's problem. Therefore, the firm's problem can be solved using QVI. However, one cannot expect that a solution of the QVI to be sufficiently continuous. To overcome the irregularity in the solution of stochastic control problems, Crandall and Lions (1983) introduce a weak solution concept called the viscosity solution. Therefore, we derive the value function of the firm's problem as a viscosity solution of the QVI, following Federico et al. (2019) who apply the viscosity solution method to solve the capital expansion problem using impulse control. Further, we numerically derive the optimal capital expansion and reduction policy by solving the simultaneous equations. Finally, we conduct a comparative static analysis to gain managerial insights under output demand ambiguity.

The remainder of this paper is organized as follows. In Section 2, we describe a firm's capital expansion and reduction problem when the output demand is ambiguous. Section 3 defines the quasi-variational inequalities to solve the firm's problem. Section 4 shows that the value function of the firm's problem is a viscosity solution of the QVI. Section 5 solves the firm's problem through QVI. We numerically derive an optimal capital expansion and reduction policy and conduct a comparative static analysis in Section 6. Finally, Section 7 concludes the paper.

2 Firm's Problem

The output demand X, for a firm that produces an output using capital K and sells it in a competitive market, is random and its process is governed by the geometric Brownian motion :

$$\mathrm{d}X_t^x = \mu X_t^x \mathrm{d}t + \sigma X_t^x \mathrm{d}W_t^{\mathbb{P}}, \ \ X_{0^-} = x > 0, \tag{2.1}$$

where $\mu > 0$ and $\sigma > 0$ are constants. $W_t^{\mathbb{P}}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. The firm controls the level of capital according to the output demand. Here, we consider a probability in which the firm's manager faces ambiguity in deciding the distribution of the output demand. We follow Chen and Epstein (2002) and Cheng and Riedel (2013), in controlling the ambiguity in output demand.

Let $\mathcal{P}$ be the set of all probability measures that are considered by the firm and $\mathbb{Q}$ be equivalent to $\mathbb{P}$ with density $\exp\{-\frac{1}{2}\int_0^t \theta_s^2 \mathrm{d}s + \int_0^t \theta_s \mathrm{d}W_s\}$:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left\{-\frac{1}{2}\int_0^t \theta_s^2 \mathrm{d}s + \int_0^t \theta_s \mathrm{d}W_s\right\}, \tag{2.2}$$

where $\theta = \{\theta_t\}_{t\geq 0}$ with $|\theta_t| \leq \kappa$ satisfies Novikov's condition :

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^t \theta_s^2 \mathrm{d}s\right\}\right] < \infty. \tag{2.3}$$

This model of ambiguity implies that the output demand process has ambiguity in its drift rate. The drift shifts between -κ and κ, in which case we express the ambiguity as κ-ignorance. Cheng and Riedel (2013) confirm that κ-ignorance can be applied in an infinite time horizon model, similar to this study. Applying Girsanov's theorem, under the probability measure $\mathbb{Q} \in \mathcal{P}$, we get

$$\mathrm{d}W_t^{\mathbb{Q}} = \mathrm{d}W_t^{\mathbb{P}} + \theta_t \mathrm{d}t \,. \tag{2.4}$$

Based on this result, the dynamics of the output demand are rewritten as

$$\begin{aligned} \mathrm{d}X_t^x &= \mu X_t^x \mathrm{d}t + \sigma X_t^x (\mathrm{d}W_t^{\mathbb{Q}} - \theta_t \mathrm{d}t) \\ &= (\mu - \theta_t \sigma) X_t^x \mathrm{d}t + \sigma X_t^x \mathrm{d}W_t^{\mathbb{Q}}. \end{aligned} \tag{2.5}$$

Let ζ_i be the ith amount of change in capital at time τ_i, $i = 0, 1, \cdots$. The process of the capital stock is governed by the following differential equation :

$$\begin{cases} \mathrm{d}K_t^k = -\delta K_t^k \mathrm{d}t, & \tau_i \le t < \tau_{i+1}, \\ K_{\tau_i}^k = K_{\tau_i^-}^k + \zeta_i, & \\ K_{0^-} = k\,(> 0), & \end{cases} \tag{2.6}$$

where $\delta \in (0, 1)$ is a constant depreciation rate. The firm's operating profit $\hat{\pi}$ at time t is specified as

$$\hat{\pi}(K_t, X_t) = K_t^{\alpha} X_t^{1-\alpha}, \tag{2.7}$$

where $\alpha \in (0, 1)$. Equation (2.7) implies that the firm has a constant-returns-to-scale Cobb-Douglas production function facing an iso-elastic demand curve (Abel and Eberly, 1996).

The firm can purchase capital at a constant unit price $p > 0$ and sell it at a constant price $(1 - \lambda)p > 0$. The purchase and sale price are the proportional costs. The parameter $\lambda \in (0, 1)$ represents the degree of irreversibility of the investment. The investment cost is completely sunk if λ becomes 1. In addition to the proportional cost, changing the level of capital stock involves a fixed cost $c > 0$. The fixed cost represents the cost of research related to decision making. To summarize the cost of changing the level of capital, the capital expansion and reduction costs are given by

$$C(\zeta_i) = \begin{cases} c + p\zeta_i, & \zeta_i > 0, \\ c, & \zeta_i = 0, \\ c + (1 - \lambda) p \zeta_i, & \zeta_i < 0. \end{cases} \tag{2.8}$$

The cost function $C(\zeta)$ satisfies the subadditivity property for all ζ, ζ' :

$$C(\zeta+\zeta') \leq C(\zeta)+C(\zeta'). \tag{2.9}$$

Inequality (2.9) implies that reasonable $\{\mathcal{F}t\}_{t\geq 0}$-stopping times become strictly increasing sequences : $\tau_0<\tau_1<\tau_2<\cdots<\tau_i<\cdots$.

The capital expansion and reduction policy $\hat{v}$ is defined as the following double sequence :

$$\hat{v} := \{(\tau_i, \zeta_i)\}_{i\geq 0}, \tag{2.10}$$

where $\zeta := \{\zeta_i\}_{i\geq 0}$ is a sequence of $\mathcal{F}\,\tau_i$-measurable random variables that indicates the ideal size for changing the level of the capital stock. Given the capital expansion and reduction policy $\hat{v}$, $\{K_t^{k,\hat{v}}\}_{t\geq 0}$, and $\{X_t^{x,\hat{v}}\}_{t\geq 0}$ represents the process of K_t and X_t with initial values $(K_{0^-}, X_{0^-}) = (k, x)$, respectively.

We assume that the firm's manager is ambiguity averse. Let $\hat{v}_0$ represent the policy under which the firm's manager does not change the level of capital, except for depreciation. We assume that the expected discounted profit under $\hat{v}_0$ satisfies

$$\begin{aligned}\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty e^{-rt}\hat{\pi}(K_t^{k,\hat{v}_0}, X_t^{x,\hat{v}_0})dt\right] &= \inf_{\theta\in[-\kappa,\kappa]} \mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty e^{-rt}\hat{\pi}(K_t^{k,\hat{v}_0}, X_t^{x,\hat{v}_0})dt\right]\\ &= \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^\infty e^{-rt}\hat{\pi}(K_t^{k,\hat{v}_0}, X_t^{x,\hat{v}_0})dt\right]\\ &= Bk^\alpha x^{1-\alpha}\\ &< \infty,\end{aligned} \tag{2.11}$$

where $\mathbb{E}_{\mathbb{Q}}$ is the expectation operator with respect to the measure $\mathbb{Q}$, $r>0$ is the discount rate, and B is defined by

$$B := \frac{1}{(r-\mu+\kappa\sigma)+(\delta+\mu-\kappa\sigma)\alpha-\frac{1}{2}\sigma^2\alpha(\alpha-1)}. \tag{2.12}$$

To hold (2.11), we assume that

$$B>0. \tag{2.13}$$

The firm's expected discounted net profit $\hat{J}(k, x; \hat{v})$ is given by

$$\hat{J}(k,x;\hat{v}) = \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\infty} \mathrm{e}^{-rt}\hat{\pi}(K_t^{k,\hat{v}}, X_t^{x,\hat{v}})\mathrm{d}t - \sum_{i=1}^{\infty} \mathrm{e}^{-r\tau_i} C(\zeta_i)\mathbf{1}_{\{\tau_i<\infty\}}\right]. \tag{2.14}$$

The infimum operator reflects the firm's manager's ambiguity aversion (Wang, 2010).

Hereinafter, for simplicity, we assume the change variables as $Y_t := K_t/X_t$. Then, the operating profit function $\hat{\pi}$, the capital expansion and reduction size ζ_i, and expected discounted net profit $\hat{J}(k,x;\hat{v})$ can be rewritten as follows :

$$\hat{\pi}(k,x) = k^{\alpha}x^{1-\alpha} = y^{\alpha}x = \pi(y)x\ ; \tag{2.15}$$

$$\zeta_i/x =: \xi_i\ ; \tag{2.16}$$

$$\hat{J}(k,x) = x\hat{J}\left(\frac{k}{x},1\right) = xJ(y). \tag{2.17}$$

We define the capital expansion and reduction policy as follows :

Definition 2.1 (Capital expansion and reduction policy). *A capital expansion and reduction policy v is composed using the capital expansion and reduction timing and its magnitude :*

$$v := \{(\tau_i,\xi_i)\}_{i\geq 0}. \tag{2.18}$$

The firm's expected discounted net profit is rewritten as follows :

$$J(y;v) = \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\infty} \mathrm{e}^{-rt}\pi(Y_t^{y,v})\mathrm{d}t - \sum_{i=1}^{\infty} \mathrm{e}^{-r\tau_i} C(\xi_i)\mathbf{1}_{\{\tau_i<\infty\}}\right]. \tag{2.19}$$

We define a set of admissible capital expansion and reduction policy as follows :

Definition 2.2 (Admissible capital expansion and reduction policy). *A capital expansion and reduction policy v is admissible if the following conditions are satisfied :*

$$0 \leq \tau_i \leq \tau_{i+1},\ a.s.\ i \geq 0\ ; \tag{2.20}$$

$$\tau_i\ \textit{is an}\ \{\mathcal{F}_t\}_{t\geq 0}\textit{-stopping time},\ i\geq 0\ ; \tag{2.21}$$

$$\xi_i\ \textit{is an}\ \mathcal{F}_{\tau_i}\textit{-measurable},\ i\geq 0\ ; \tag{2.22}$$

$$P\left[\lim_{i\to\infty}\tau_i \leq \hat{T}\right] = 0,\ \ \hat{T}\in[0,\infty). \tag{2.23}$$

Condition (2.23) implies that the capital expansion and reduction policy will only occur finitely before a terminal time, $\hat{T}$. Let $\mathcal{V}$ be a set of admissible capital expansion and reduction policy. Therefore, the firm's problem is given as follows :

Problem 2.1 (Firm's problem). *The firm's problem is to maximize the expected discounted net profit J over* $\mathcal{V}$.

$$V(y) = \sup_{v \in \mathcal{V}} J(y; v) = J(y; v^*), \tag{2.24}$$

where V is the value function and v^* is the optimal capital expansion and reduction policy.

The firm's problem (2.24) is formulated as a stochastic impulse control problem.

Proposition 2.1 *The firm's expected discounted net profit function $J(y; v)$ is well defined and finite if the following conditions hold in addition to (2.13) :*

$$\lim_{t \to \infty} \mathbb{E}_{\mathbb{Q}} \left[e^{-rt} Y_t^{y,v} \right] = 0. \tag{2.25}$$

Proof. If condition (2.23) holds, then we obtain

$$\mathbb{E}_{\mathbb{Q}} \left[\sum_{i=1}^{\infty} e^{-r\tau_i} \mathbf{1}_{\{\tau_i < \infty\}} \right] < \infty. \tag{2.26}$$

According to the integration by parts formula, for every $0 < s \leq t < \infty$ (Rogers and Williams, 2000, VI 38), we have

$$\mathbb{E}_{\mathbb{Q}}[e^{-rt} Y_t] - \mathbb{E}_{\mathbb{Q}}[e^{-rs} Y_s] = -(r + \delta + (\mu - \theta\sigma) - \sigma^2)\, \mathbb{E}_{\mathbb{Q}} \left[\int_s^t e^{-ru} Y_t \, du \right] \\ + \mathbb{E}_{\mathbb{Q}} \left[\sum_{i=1}^{\infty} e^{-r\tau_i} \xi_i \mathbf{1}_{\{s < \tau_i \leq t\}} \right]. \tag{2.27}$$

Using (2.25), we obtain

$$\mathbb{E}_{\mathbb{Q}} \left[\sum_{i=1}^{\infty} e^{-r\tau_i} \xi_i \mathbf{1}_{\{s < \tau_i \leq t\}} \right] < \infty. \tag{2.28}$$

It is confirmed from (2.26) and (2.28) that the expected discounted cost is finite :

$$\mathbb{E}_{\mathbb{Q}} \left[\sum_{i=1}^{\infty} e^{-r\tau_i} C(\xi_i) \mathbf{1}_{\{\tau_i < \infty\}} \right] < \infty. \tag{2.29}$$

Therefore, the firm's expected discounted net profit function, J, is well defined and finite.

□

3 Quasi-variational Inequalities for the Firm's Problem

The formulation of the firm's problem (2.24) enables us to speculate an optimal capital expansion and reduction policy defined by thresholds. Once the output demand reaches a certain level, the firm immediately expands or reduces its capital. Consequently, the level of capital is maintained within the capital expansion or reduction threshold. To verify this conjecture, we prove that a capital expansion and reduction policy induced by QVI is optimal.

The capital expansion and reduction operator $\mathcal{M}$ can be defined as

$$\mathcal{M}V(y) = \sup_{\xi \in \mathbb{R}, y+\xi \in \mathbb{R}_{++}} \{V(y+\xi) - C(\xi)\}. \tag{3.1}$$

$\mathcal{M}V(y)$ represents the value of the policy that involves choosing the optimum level of change in capital. It is not optimal to change the level of capital for all y. Thus, we get the following inequality :

$$V(y) \geq \mathcal{M}V(y). \tag{3.2}$$

In contrast, if it is optimal to change the level of capital for y, then the $V(y)$ and $\mathcal{M}V(y)$ must coincide.

Furthermore, we introduce the differential operator $\mathcal{L}$:

$$\mathcal{L}V(y) := \frac{1}{2}\sigma^2 y^2 V''(y) - (\delta + \mu - \kappa\sigma) y V'(y) - (r - \mu + \kappa\sigma) V(y). \tag{3.3}$$

To derive the operator $\mathcal{L}$, it should be noted that the derivatives of the firm's expected discounted profit function are calculated as $\hat{J}_K(k,x) = J'(y)$, $\hat{J}_X(k,x) = J(y) - yJ'(y)$, and $\hat{J}_{XX}(k,x) = (y^2/x)J''(y)$. Then, the QVI of the firm's problem is defined as follows :

Definition 3.1 (QVI).*Considering $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ as a function, the following three relations would be the QVI for the firm's problem :*

$$\mathcal{L}\phi(y) + \pi(y) \leq 0 ; \tag{3.4}$$

$$\phi(y) \geq \mathcal{M}\phi(y) ; \tag{3.5}$$

$$[\mathcal{L}\phi(y) + \pi(y)][\mathcal{M}\phi(y) - \phi(y)] = 0. \tag{3.6}$$

The QVI is summarized as

$$\max\{\mathcal{L}\phi(y)+\pi(y),\ \mathcal{M}\phi(y)-\phi(y)\}=0. \tag{3.7}$$

The definition of the QVI enables us to divide the interval $(0,\infty)$ into three regions : the continuation region $\mathcal{H}$, capital expansion region $\mathcal{E}$, and capital reduction region $\mathcal{R}$. They are given as follows :

$$\mathcal{H}:=\{y\in(0,\infty);\phi(y)>\mathcal{M}\phi(y)\text{ and }\mathcal{L}V(y)+\pi(y)=0\}\ ; \tag{3.8}$$

$$\mathcal{E}:=\{y\in(0,\infty);\phi(y)=\mathcal{M}\phi(y)\text{ and }\mathcal{L}V(y)+\pi(y)=0,\ \xi>0\}\ ; \tag{3.9}$$

$$\mathcal{R}:=\{y\in(0,\infty);\phi(y)=\mathcal{M}\phi(y)\text{ and }\mathcal{L}V(y)+\pi(y)=0,\ \xi<0\}. \tag{3.10}$$

We define a policy that is derived from the QVI as follows :

Definition 3.2 (QVI policy). *Let ϕ be a solution to QVI (3.4) - (3.6). Then, the following capital expansion and reduction policy $\tilde{v}=\{\tilde{\tau}_i,\tilde{\xi}_i\}_{i\geq 0}$ would be the QVI policy :*

$$(\tilde{\tau}_0,\tilde{\xi}_0)=(0,0)\ ; \tag{3.11}$$

$$\tilde{\tau}_i=\inf\{t\geq\tilde{\tau}_{i-1};Y_t^{y,\tilde{v}}\notin\mathcal{H}\}\ ; \tag{3.12}$$

$$\tilde{\xi}_i=\arg\max_{\xi}\left\{\phi\left(Y_{\tilde{\tau}_i^-}^{y,\tilde{v}}+\xi_i\right)-C(\xi_i);\xi_i\in\mathbb{R},Y_{\tilde{\tau}_i^-}^{y,\tilde{v}}+\xi_i\in\mathbb{R}_{++}\right\}. \tag{3.13}$$

The QVI policy assumes that once ϕ and $\mathcal{M}\phi$ coincide, the firm changes the level of capital by using a QVI policy. We verify that the QVI policy is the optimal capital expansion and reduction policy, as proposed by Tsujimura (2020, Theorem 3.1).

Theorem 3.1 *(I) Suppose that ϕ is in $C^1(\mathbb{R}_{++})$ and $C^2(\mathbb{R}_{++}\setminus\mathcal{N})$, where $\mathcal{N}\subset\mathbb{R}_{++}$. Suppose that there exists $0<\underline{y}<\overline{y}<\infty$ such that ϕ is linear in $y\in(0,\underline{y}]\cup[\overline{y},\infty)$. We assume that the family $\{\phi(Y_\tau^{y,v})\}_{\tau<\infty}$ is uniformly integrable for all $y\in\mathbb{R}_{++}$ and $v\in\mathcal{V}$. If a solution ϕ of the QVI to the firm's problem (2.24) exists, then, for all $y\in\mathbb{R}_{++}$, we obtain*

$$\phi(y)\geq V(y). \tag{3.14}$$

(II) If the QVI policy corresponding to ϕ is admissible, $\tilde{v}\in V$, then, for all $y\in\mathbb{R}_{++}$, ϕ is the value function

$$\phi(y) = V(y),$$

and $\tilde{v}$ *is optimal.*

Proof. See Proof of Theorem 3.1 of Tsujimura (2020) □

4 Viscosity Solutions of the Firm's Problem

It is not generally expected that a solution of the QVI, ϕ, is C^2 for all y in Theorem 3.1 (Korn, 1999, Remark 2.6). We assume this to be true for a one-dimensional impulse control problem. ϕ is C^2 within a continuation region and linear otherwise, according to the Theorem 3.1. This irregularity is solved using the viscosity solution introduced by Crandall and Lions (1983). See also Crandall et al.(1992), Reikvam (1998), Øksendal and Sulem (2002), Guan and Liang (2014), and Federico et al. (2019) for more details about viscosity solutions.

We introduce a viscosity solution to solve the firm's problem.

Definition 4.1 (Viscosity solution). *Let* $\phi : \mathbb{R}_{++} \to \mathbb{R}_{+}$ *be a Lipschitz continuous function such that lim* $\sup_{y\to\infty}\phi(y)/y < p$.

(i) ϕ *is a* viscosity supersolution *to the QVI if for all* $(y_0, \varphi) \in \mathbb{R}_{++} \times C^2(\mathbb{R}_{++})$*, such that* $\phi - \varphi$ *has a local minimum at* y_0 *and* $\phi(y_0) = \varphi(y_0)$*, therefore,*

$$\min\{-(\mathcal{L}\varphi(y_0) + \pi(y_0)),\ \ \phi(y_0) - \mathcal{M}\phi(y_0)\} \geq 0. \tag{4.1}$$

(ii) ϕ *is a* viscosity subsolution *to the QVI if for all* $(y_0, \varphi) \in \mathbb{R}_{++} \times C^2(\mathbb{R}_{++})$*, such that* $\phi - \varphi$ *has a local maximum at* y_0 *and* $\phi(y_0) = \varphi(y_0)$*, then we have*

$$\min\{-(\mathcal{L}\varphi(y_0) + \pi(y_0)),\ \ \phi(y_0) - \mathcal{M}\phi(y_0)\} \leq 0. \tag{4.2}$$

(iii) ϕ *is a* viscosity solution *to the QVI, if it is both a viscosity sub- and supersolution.*

We propose that the value function of the firm's problem (2.24) is a viscosity solution, as proposed by Federico et al. (2019).irreversible}.

Proposition 4.1 *The value function of the firm's problem (2.24}) is the viscosity solution of the QVI.*

Proof. (a) We first verify the viscosity supersolution property. Let $y_0 \in \mathbb{R}_{++}$ and $\varphi \in C^2(\mathbb{R}_{++})$ be such that $V - \varphi$ has a local minimum at y_0 and $V(y_0) = \varphi(y_0)$. For a suitable $\varepsilon \in (0, y_0)$, we have $V \geq \varphi$ on $(y_0 - \varepsilon, y_0 + \varepsilon)$. It is understood from the definition of the capital expansion and reduction operator $\mathcal{M}$ that we have $V(y) \geq \mathcal{M}V(y)$. Therefore, it is only required to show that $-(\mathcal{L}\varphi(y_0) + \pi(y_0)) \geq 0$. Accordingly, let $\tau' := \{t \geq 0; |Y_t^{y_0,v_0} - y_0| > \varepsilon\}$ be a stopping time. It is to be noted that $\mathbb{Q}\{\tau' > 0\} = 1$ due to the continuity of trajectories of $Y_t^{y_0,v_0}$. Applying the dynamic programming principle, we obtain the following for $\varpi > 0$

$$V(y_0) \geq \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau' \wedge \varpi} \mathrm{e}^{-rt}\pi(Y_t^{y_0,v_0})\mathrm{d}t + \mathrm{e}^{-r(\tau' \wedge \varpi)}V\left(Y_{\tau' \wedge \varpi}^{y_0,v_0}\right)\right]. \tag{4.3}$$

$V \geq \varphi$on$(y_0 - \varepsilon, y_0 + \varepsilon)$ yields the following inequality

$$V(y_0) \geq \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau' \wedge \varpi} \mathrm{e}^{-rt}\pi(Y_t^{y_0,v_0})\mathrm{d}t + \mathrm{e}^{-r(\tau' \wedge \varpi)}\varphi(Y_{\tau' \wedge \varpi}^{y_0,v_0})\right] \tag{4.4}$$

Applying Dynkin's formula, we obtain

$$V(y_0) \geq \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau' \wedge \varpi} \mathrm{e}^{-rt}\pi(Y_t^{y_0,v_0})\mathrm{d}t + \left(\varphi(y_0) + \int_0^{\tau' \wedge \varpi} \mathrm{e}^{-rt}\mathcal{L}\varphi(Y_t^{y_0,v_0})\mathrm{d}t\right)\right]. \tag{4.5}$$

Arranging (4.5) leads to the following inequality

$$\mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau' \wedge \varpi} \mathrm{e}^{-rt}\left(\pi(Y_t^{y_0,v_0}) + \mathcal{L}\varphi(Y_t^{y_0,v_0})\right)\mathrm{d}t\right] \leq 0. \tag{4.6}$$

Taking $\lim_{\varpi \to 0^+}$ by using the dominated convergence theorem and $\mathbb{Q}\{\tau' > \varpi\} \to 1$ as $\varpi \to 0^+$, we obtain

$$-(\mathcal{L}\varphi(y_0) + \pi(y_0)) \geq 0. \tag{4.7}$$

Thus, we have proved the viscosity supersolution property.

(b) Next, we verify the viscosity subsolution property. Let $y_0 \in \mathbb{R}_{++}$ and $\varphi \in C^2(\mathbb{R}_{++})$ be such that $V - \varphi$ has a local maximum at y_0 and $V(y_0) = \varphi(y_0)$. By $V(y) \geq \mathcal{M}V(y)$, we assume that $V(y_0) \geq \mathcal{M}V(y_0) + \varsigma$ for some $\varsigma > 0$. It is only required to verify that $-(\mathcal{L}\varphi(y_0) + \pi(y_0)) \leq 0$. We prove this by contradiction. We assume that $-(\mathcal{L}\varphi(y_0) + \pi(y_0)) \geq \varsigma > 0$. By the continuity of $-(\mathcal{L}\varphi(y_0) + \pi(y_0))$ and $V(y) - \mathcal{M}V(y)$, there exists $\varepsilon \in (0, y_0/2)$, such that for all y in a neighborhood of y_0, $B(y_0, 2\varepsilon]$:

$$-(\mathcal{L}\varphi(y_0) + \pi(y_0)) \geq \varpi/2, \tag{4.8 a}$$

$$\varphi(y) \geq V(y), \tag{4.8 b}$$

$$V(y) - \mathcal{M}V(y) \geq \varpi/2. \tag{4.8 c}$$

From (4.8 c), it is not optimal to change the level of capital for $y \in B(y_0, 2\varepsilon]$. Hence, the dynamic programming principle can be rewritten such that the case in which the first capital changing time τ_1 is later than τ' :

$$V(y_0) = \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau'} \mathrm{e}^{-rt}\pi(Y_t^{y_0,v_0})\mathrm{d}t + \mathrm{e}^{-r\tau'}V(Y_{\tau'}^{y_0,v_0})\right]. \tag{4.9}$$

Based on (4.8 a), we have

$$\frac{\varpi}{2}\mathbb{E}_{\mathbb{Q}^\kappa}[\tau'] \leq \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau'} \mathrm{e}^{-rt}(-(\mathcal{L}\varphi(Y_t^{y_0,v_0}) + \pi(Y_t^{y_0,v_0})))\mathrm{d}t\right]. \tag{4.10}$$

Applying Dynkin's formula to (4.10), we get

$$\frac{\varpi}{2}\mathbb{E}_{\mathbb{Q}^\kappa}[\tau'] \leq \varphi(y_0) - \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau'} \mathrm{e}^{-rt}\pi(Y_t^{y_0,v_0}))\mathrm{d}t + e^{-r\tau'}\varphi(Y_{\tau'}^{y_0,v_0})\right]. \tag{4.11}$$

Based on (4.8 b) and $V(y_0) = \varphi(y_0)$, we obtain

$$\begin{aligned}\frac{\varpi}{2}\mathbb{E}_{\mathbb{Q}^\kappa}[\tau'] &\leq \varphi(y_0) - \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau'} \mathrm{e}^{-rt}\pi(Y_t^{y_0,v_0}))\mathrm{d}t + e^{-r\tau'}\varphi(Y_{\tau'}^{y_0,v_0})\right] \\ &= V(y_0) - \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau'} \mathrm{e}^{-rt}\pi(Y_t^{y_0,v_0}))\mathrm{d}t + \mathrm{e}^{-r\tau'}\varphi(Y_{\tau'}^{y_0,v_0})\right], \\ &\leq V(y_0) - \mathbb{E}_{\mathbb{Q}^\kappa}\left[\int_0^{\tau'} \mathrm{e}^{-rt}\pi(Y_t^{y_0,v_0}))\mathrm{d}t + e^{-r\tau'}V(Y_{\tau'}^{y_0,v_0})\right], \\ &= 0\end{aligned} \tag{4.12}$$

The definition of τ' yields $\mathbb{Q}\{\tau' > 0\} = 1$, however, the above result contradicts it. This proves the viscosity subsolution property. □

5 Solution of the Quasi-variational Inequalities of the Firm's Problem

We assume that an optimal capital expansion and reduction policy $v^* \in \mathcal{V}$ is characterized by four thresholds : y_E, y_e, y_r, and y_R with $0 < y_E < y_e < y_r < y_R < \infty$. Once the level of Y reaches y_E or y_R, the firm purchases or sells the capital, and the level of Y instantaneously increases or decreases to another level of Y, y_e or y_r, respectively. Consequently, the level of Y changes by $y_e - y_E$ or $y_r - y_R$ at each time τ_i.

Based on the assumption above, we can define the optimal capital expansion and reduction policy $v^* = (\tau^*, \xi^*) \in \mathcal{V}$ such that

$$\tau_i^* := \inf\{t > \tau_{i-1}^*; Y_{t^-} \notin (y_E, y_R)\} \; ; \tag{5.1}$$

$$\xi_i^* := Y_{\tau_i} - Y_{\tau_i^-} = \begin{cases} y_e - y_E, & Y_{\tau_i^-} = y_E, \\ y_r - y_R, & Y_{\tau_i^-} = y_R. \end{cases} \tag{5.2}$$

The definition of v^* resets the three regions : the continuation region, capital expansion region, and capital reduction region as follows :

$$\mathcal{H} := \{y; y_E < y < y_R\}, \quad \mathcal{E} := \{y; y \leq y_E\}, \quad \mathcal{R} := \{y; y \geq y_R\}. \tag{5.3}$$

For $y \in \mathcal{H}$, QVI (3.4) - (3.6) leads to the following ordinary differential equation (ODE) :

$$\mathcal{L}\phi(y) + \pi(y) = 0. \tag{5.4}$$

The general solution of the ODE (5.4) is given by

$$\phi(y) = A_1 y^{\gamma_1} + A_2 y^{\gamma_2} + B y^{\alpha}, \quad y \in \mathcal{H}, \tag{5.5}$$

where A_1 and A_2 are constants to be determined, and γ_1 and γ_2 are the solutions to the following characteristic equation :

$$\frac{1}{2}\sigma^2\gamma^2 - \left(\delta + \mu - \kappa\sigma + \frac{1}{2}\sigma^2\right)\gamma - (r - \mu) = 0. \tag{5.6}$$

γ_1 and γ_2 are calculated with

$$\gamma_1 = \frac{\delta + \mu - \kappa\sigma}{\sigma^2} + \frac{1}{2} + \left[\left(\frac{\delta + \mu - \kappa\sigma}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2(r - \mu + \kappa\sigma)}{\sigma^2}\right]^{\frac{1}{2}} > 1 \ ;$$

$$\gamma_2 = \frac{\delta + \mu - \kappa\sigma}{\sigma^2} + \frac{1}{2} - \left[\left(\frac{\delta + \mu - \kappa\sigma}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2(r - \mu + \kappa\sigma)}{\sigma^2}\right]^{\frac{1}{2}} < 0. \tag{5.7}$$

The parameter B is given by (2.12). We assume that the firm has the option to expand the capital or reduce it. A higher value of Y means a smaller demand for the output and/or excess of capital. Accordingly, the first term of (5.5) represents the option to reduce the capital. In contrast, a lower value of Y means a higher demand for output and/or inadequate capital. Therefore, the second term of (5.5) represents the option to expand the capital. The positivity of the option values implies that both the constants A_1 and A_2 must be positive.

When $y \in \mathcal{E}$, the firm expands its capital and the level of Y increases to y_e. Then, the following equation holds :

$$\phi(y) = \phi(y_e) - (c + p(y_e - y)), \ \ y \in \mathcal{E}. \tag{5.8}$$

In contrast, for $y \in \mathcal{R}$, the firm reduces the capital, and the level of Y decreases to y_r. Then, the following equation holds :

$$\phi(y) = \phi(y_r) - (c + (1 - \lambda)p(y_r - y)), \ \ y \in \mathcal{R}. \tag{5.9}$$

If the candidate function of the value function is differentiable in $\{y_E, y_R\}$, from equations (5.8) and (5.9) we obtain the following equations :

$$\phi'(y_E) = p \ ; \tag{5.10}$$

$$\phi'(y_R) = (1 - \lambda)p. \tag{5.11}$$

Equation (5.10) implies that the marginal value of the net profit is equal to the unit purchase price of the capital when the firm expands its capital. Similarly, (5.11) implies that the marginal value of the net profit is equal to the unit sale price of the capital when the firm reduces its capital. Based on the definition of the optimal capital expansion and reduction policy (5.1) and (5.2), the firm's expected discounted net profit J is maximized at $\xi = y_e - y_E$ or $\xi = y_r - y_R$. Hence, by the first-order condition for the maximization

$\mathrm{d}[\phi(y_E+\xi)-C(\xi)]/\mathrm{d}\xi|_{\xi=y_e-y_E}=0$ or $\mathrm{d}[\phi(y_R+\xi)-C(\xi)]/\mathrm{d}\xi|_{\xi=y_r-y_R}=0$. If $\phi(y)$ are differentiable in $\{y_e, y_r\}$, we obtain

$$\phi'(y_e)=p\ ; \tag{5.12}$$

$$\phi'(y_r)=(1-\lambda)p. \tag{5.13}$$

Consequently, we assume that the optimal solutions described by (5.1) and (5.2) and the six unknowns, A_1, A_2, y_E, y_e, y_r, y_R, are a solution to the simultaneous equations :

$$\phi(y_E)=\phi(y_e)-(c+p(y_e-y_E)), \tag{5.14}$$

$$\phi(y_R)=\phi(y_r)-(c+(1-\lambda)p(y_r-y_R)), \tag{5.15}$$

and (5.10) - (5.13), where ϕ is the solution of the ODE (5.4) for $y\in\mathcal{H}$.

6 Numerical Analysis

In this section, we conduct a numerical analysis to obtain insights that could benefit the firm's decision-making. To further clarify the effects of ambiguity, we use parameter values of Tsujimura (2021), except κ : $r=0.05$, $\delta=0.1$, $\mu=0.01$, $\sigma=0.15$, $\alpha=0.6$, $c=1$, $p=10$, and $\lambda=0.5$. In addition to these parameter values, we set $\kappa=0.01$ as a base case.

Solving the simultaneous equations (5.10) - (5.15) numerically, we obtain $A_1=3.26248*10^{-6}$, $A_2=0.0451602$, $y_E=0.00221966$, $y_e=0.216875$, $y_r=1.25618$, and $y_R=2.88707$. Figure 1 illustrates the value function of the firm's problem (2.24) based on the base case parameter values. Figure 2 shows that the value function for $y\in(0,\ 0.01)$ as the capital expansion threshold y_E is very low.

Table 1 summarizes the results of the comparative statics. We focus on the results of the numerical analysis of the ambiguity parameter κ, risk parameter σ, fixed cost parameter c, proportional cost parameter p, and irreversible parameter λ. First, the results of the output demand ambiguity parameter κ show that the capital expansion threshold y_E and the capital reduction threshold y_R decrease with increasing κ. These results mean that the capital expansion is delayed as the output demand ambiguity increases, while the capital reduction is promoted. Combining these effects decreases the continuation region $\mathcal{H}$ in the output

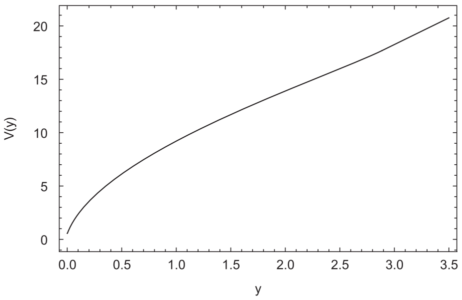

Figure 1　The value function of the firm's problem.

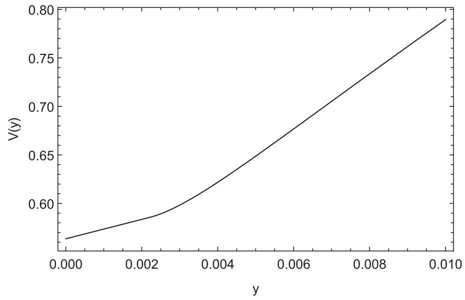

Figure 2　The value function of the firm's problem for $y \in (0, 0.01)$.

Table 1　Comparative static results

	A_1	A_2	y_E	y_e	y_r	y_R	$\mathcal{H}$	$\xi_i\,(>0)$	$\lvert\xi_i\,(<0)\rvert$
κ	+	−	−	−	−	−	−	−	−
r	+	−	−	−	−	−	−	−	−
μ	−	+	+	+	+	+	+	+	+
σ	+	−	−	−	−	+	+	−	+
δ	−	−	−	−	−	−	−	−	−
α	−	−	−	−	+	+	+	−	+
c	−	−	−	+	−	+	+	+	+
p	+	−	−	−	−	−	−	−	−
λ	−	n.c.	n.c.	n.c.	+	+	+	n.c.	+

"n.c." means the value does not change if the associated parameter value is varied.

ambiguity. This exhibits the firm's manager's precautionary behavior in deciding the capital investment under output demand ambiguity. With respect to the extent of change in the capital, ξ_i, it decreases as the output demand ambiguity increases. Furthermore, if there is no output demand ambiguity, i.e., when $\kappa = 0$, the current model reduces to the output demand risk model (Tsujimura, 2021). Compared to the results of the numerical analysis in Tsujimura (2021), all thresholds y_E, y_e, y_r, and y_R in the base case decreased by 3.83%, 1.08%, 1.29%, and 0.8%, respectively. Consequently, the continuation region $\mathcal{H}$, the size of capital expansion, and capital reduction decrease by 0.8%, 1.05%, and 0.42%, respectively.

Second, the results of the output demand risk parameter σ show that the capital expansion threshold y_E decreases with increasing σ, while the capital reduction threshold y_R increases with σ. These results indicate that both the capital expansion and reduction are delayed as the output demand risk increases. The result of the continuation region $\mathcal{H}$ secures the effect of the output demand risk. Regarding the extent of change in the capital, the size of capital expansion decreases with increasing the output demand risk, while the size of capital reduction increases.

Third, the comparative statics to the fixed cost c show that the capital expansion threshold

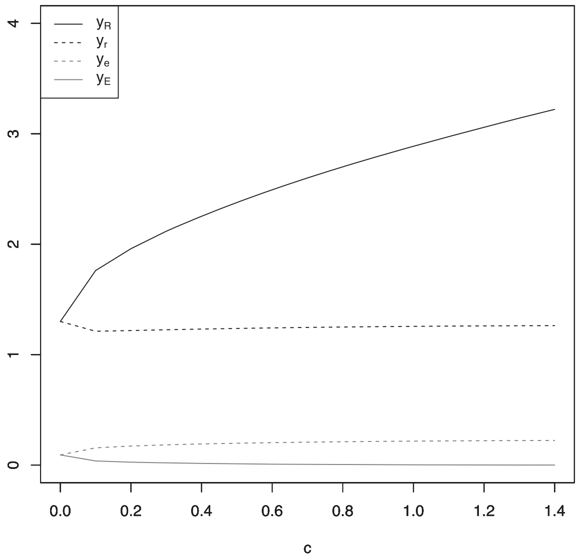

Figure 3 Effect of the changes in the fixed cost, c, on the four thresholds

y_E decreases with increasing c, while the capital reduction threshold y_R increases with c in the same way as σ. These results indicate that both the capital expansion and reduction are postponed as the fixed cost increases. The continuation region $\mathcal{H}$ shows the effect of the fixed cost on the capital level change timing. For each new timing, the extent of change in the capital increases with the fixed cost. In addition to the comparative static analyses, Figure 3 shows that each threshold in both pairs of thresholds (y_E, y_e) and (y_r, y_R) converges to a certain level, which represents the capital expansion threshold (lower threshold) and the capital reduction threshold (higher threshold) as the fixed cost becomes 0. In the absence of fixed cost, the firm's problem would be formulated as a singular stochastic control problem. Once the level of Y reaches the threshold, the firm expands or reduces the capital so that the level of Y does not exceed the thresholds. See Tsujimura (2019) for more details.

Fourth, the comparative statics to the proportional cost parameter p shows that both the capital expansion threshold y_E and the capital reduction threshold y_R decrease with increasing p. This means that a higher value of the proportional cost parameter p discourages capital expansion while encouraging capital reduction. Combining these effects shows that the continuation region $\mathcal{H}$ decreases with increasing p. For each capital expansion or reduction timing, their magnitude decreases with increasing p.

Fifth, the results of the irreversible parameter λ influence only the capital reduction. The capital reduction threshold y_R increases with λ. This means that a higher irreversibility parameter delays the timing of capital reduction. Furthermore, the size of the capital reduction increases with λ.

7 Conclusion

In this study, we examine a firm's capital expansion and reduction problem when the output demand is ambiguous, which is described by the κ-ignorance approach. We assume that the firm has two options： the option to expand the capital and the option to reduce the capital according to the output demand. The firm can exercise the options as many times as required. Once the firm exercises an option, it incurs both fixed and proportional costs. These characteristics lead to the formulation of the firm's problem as a stochastic impulse control problem. We solve the firm's problem using QVI and show that the value function of the firm's problem is the viscosity solution. Furthermore, we numerically derive the optimal capital expansion and reduction policy and conduct comparative statics. The key finding is that the output ambiguity reduces the continuation region, whereas the output risk increases the region.

This study can be extended in various ways. First, we assume that the firm's manager can be assigned a large value of κ according to the degree of ambiguity aversion. We could include the penalty to extend the original distribution as done by Hansen and Sargent (2001) and investigate the firm's problem as a robust control problem. Second, the output demand is assumed to be governed by the geometric Brownian motion. We could extend the dynamics to a general diffusion process, such as a jump diffusion process, to describe the discrete variation of economy. Finally, the current study only considered ambiguity in the output demand. The ambiguity in the unit price of capital could also be included to further the research.

References

Abel, A. B., Eberly, J. C., 1996. Optimal investment with costly reversibility. The Review of Economic Studies 63(4), 581-593.

Chen, Z., Epstein, L., 2002. Ambiguity, risk, and asset returns in continuous time. Econometrica 70(4), 1403-1443.

Cheng, X., Riedel, F., 2013. Optimal stopping under ambiguity in continuous time. Mathematics and Financial Economics 7(1), 29-68.

Crandall, M. G., Ishii, H., Lions, P.-L., 1992. User's guide to viscosity solutions of second order partial diff erential equations. Bulletin of the American mathematical society 27(1), 1-67.

Crandall, M. G., Lions, P.-L., 1983. Viscosity solutions of Hamilton-Jacobi equations. Transactions of the American Mathematical Society 277(1), 1-42.

Ellsberg, D., 1961. Risk, ambiguity, and the savage axioms. Quarterly Journal of Economics 75(4), 643-669.

Federico, S., Rosestolato, M., Tacconi, E., 2019. Irreversible investment with fixed adjustment costs： a stochastic impulse control approach. Mathematics and Financial Economics 13(4), 579-616.

Gilboa, I., 1987. Expected utility with purely subjective non-additive probabilities. Journal of Mathematical

Economics 16(1), 65-88.

Gilboa, I., Schmeidler, D., 1989. Maximin expected utility with non-unique priors. Journal of Mathematical Economics 18, 141-153.

Guan, H., Liang, Z., 2014. Viscosity solution and impulse control of the diffusion model with reinsurance and fixed transaction costs. Insurance : Mathematics and Economics 54, 109-122.

Hansen, L., Sargent, T. J., 2001. Robust control and model uncertainty. American Economic Review 91(2), 60-66.

Hellmann, T., Thijssen, J. J. J., 2018. Fear of the market or fear of the competitor? ambiguity in a real options game. Operations Research 66(6), 1744-1759.

Knight, F. H., 1921. Risk, Uncertainty, and Profit. Houghton Miffin, Boston, MA.

Korn, R., 1999. Some applications of impulse control in mathematical finance. Mathematical Methods of Operations Research 50(3), 493-518.

Nishimura, K. G., Ozaki, H., 2007. Irreversible investment and knightian uncertainty. Journal of Economic Theory 136(1), 668-694.

Oksendal, B., Sulem, A., 2002. Optimal consumption and portfolio with both fixed and propor- tional transaction costs. SIAM Journal on Control and Optimization 40(6), 1765-1790.

Reikvam, K., 1998. Viscosity solutions of optimal stopping problems. Stochastics and Stochastic Reports 62(3-4), 285-301.

Rogers, L. C. G., Williams, D., 2000. Diffusions, Markov Processes and Martingales, 2nd Edition. Vol. 2 of Cambridge Mathematical Library. Cambridge University Press.

Sarkar, S., 2021. The uncertainty-investment relationship with endogenous capacity. Omega 98, 102115.

Schmeidler, D., 1989. Subjective probability and expected utility without additivity. Econometrica 57(3), 571-587.

Schröder, D., 2011. Investment under ambiguity with the best and worst in mind. Mathematics and Financial Economics 4(2), 107-133.

Seydel, R. C., 2009. Existence and uniqueness of viscosity solutions for qvi associated with impulse control of jump-diffusions. Stochastic Processes and their Applications 119(10), 3719- 3748.

Trojanowska, M., Kort, P. M., 2010. The worst case for real options. Journal of Optimization Theory and Applications 146(3), 709-734.

Tsujimura, M., 2019. Partially reversible capital investment under demand ambiguity. RIMS Kôkyûroku 2106, 33 -47.

Tsujimura, M., 2020. Partially reversible capital investment with both fixed and proportional costs under demand risk. RIMS Kôkyûroku 2173, 108-124.

Wang, Z., 2010. Irreversible investment of the risk-and uncertainty-averse dm under k-ignorance : The role of bsde. Annals of Economics and Finance 11(2), 313-335.