Pollution Reduction Policies and Their Associated Costs Under Uncertainty*

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In this paper, we investigate pollutant reduction policies under uncertainty. We consider two kinds of policies as distinguished by their associated costs. The first policy incurs a proportional reduction cost, while the second policy incurs fixed and proportional reduction costs. We formulate the policymaker's decision as a singular stochastic control problem in the first case and as a stochastic impulse control problem in the second case. We then derive each optimal pollution reduction policy as characterized by the thresholds required to invest in the projects. We derive the threshold of the first case explicitly, whereas we derive the threshold for the second case numerically. In addition, we employ comparative static analysis to investigate the effect of changes in the parameter values on the thresholds. First, we find that the thresholds for both cases increase in the discount rate, the uncertainty parameter, and both fixed and proportional cost parameters. Second, the thresholds for both cases decrease with the drift parameter. Finally, we find that the impact of the damage parameter in the first case depends on the level of proportional cost. The results obtained offer useful guidance for the implementation of environmental policy.

Keywords: Pollution reduction; proportional and fixed costs; singular stochastic control; stochastic impulse control

1 Introduction

The world faces many environmental problems, including climate change, acid rain, desertification, and soil contamination among others. Further, the population experiences harm, both directly and indirectly, because of these problems, which have arisen mainly as a result of environmental pollutants discharged as the by-products of economic activity. In response, there must be appropriate control of the pollutants responsible, that is, pollution targets must be set (Perman et al., 2003). This paper investigates how decision makers can reduce pollution as a means of decreasing the damage associated with pollution.

Three important characteristics of most environmental problems are uncertainty, irreversibility, and the feasibility of postponing decisions (Pindyck, 2000, 2007). First, there is uncertainty over future economic activities that discharge pollutants. This implies uncertainty concerning the future costs and benefits of environmental damage and its reduction. Second, the cost of implementing an environmental policy is a sunk cost. This typically means that environmental policy is irreversible once implemented. Environmental damage is also generally irreversible, either in part or as a whole. Finally, it is often feasible to delay the adoption of environmental policy and wait for new information. A real options approach enables us to solve environmental problems given these characteristics (Pindyck, 2000, 2002; Lin et al., 2007).

In the literature on environmental preservation, Arrow and Fisher (1974) and Henry (1974) show the value of flexibility in decision making under uncertainty[1)]. The real options approach captures the value of filexibility in investment decisions and has been applied to investigate the timing of implementing environmental policy under uncertainty (Conrad 1997; Pindyck, 2000, 2002, Saphores, 2004, Balikcioglu et al., 2011). Many researchers in the field investigate the timing of environmental policy designed to reduce a pollutant just once.

In order to describe reality more accurately, some researchers have extended this body of work. For instance, Wirl (2006) considers an agent that can both stop and restart any equipment discharging pollutants, while Lin and Huang (2010) explores the firm's cyclical investment problem, including the installation and periodic replacement of energy-saving equipment. The present paper applies two types of stochastic controls to the agent's sequential environmental investment problem in this regard where we derive the duration of investment in a project endogenously. In addition to the timing of pollution reduction, we also derive the size of the reduction endogenously. Furthermore, this paper explores how differences in the structure of investment costs affect environmental policy.

This paper provides a very general and tractable real options model for pollutant reduction policies (PRPs) under uncertainty. Suppose that an agent suffers from a pollutant emitted through an economic activity. The agent then considers investing in a pollutant reduction project to mitigate this damage. In this paper, we investigate two pollutant reduction projects. The first type of project involves investment expenditure that depends only on the size of the project, such that we can use the magnitude of the project to measure the extent to which the agent reduces the pollutant. We refer to the cost of this project as the proportional cost and to this type of investment as Case 1. For simplicity, we assume the proportional cost is constant. For example, a Case 1 project aimed at reducing greenhouse gases could involve installing a device that cuts the use of liquid fossil fuel.

In contrast, the second type of project incurs an additional cost to the proportional cost that is independent of the magnitude of the project. We refer to this independent cost as the fixed cost and to the second type of project investment as Case 2. Once again, we assume the fixed cost is constant, where the fixed cost represents costs such as research costs. In Case 2, project investment requires detailed analysis. For example, suppose that the agent invests in a carbon dioxide capture and storage (CCS) project to reduce carbon dioxide emissions, involving the capture and transfer of CO_2 emissions to a storage site, such as a deep ocean or geological site. To keep the CO_2 stored safely, the agent must find a suitable storage site, and this involves an expense for the firm (IPCC, 2005; Leung et al., 2014).

We assume that the agent does not undertake both projects simultaneously. Instead, the agent has either project available as a means of reducing the emission of pollutants, but not both[2)]. To solve these problems, in Case 1 we first formulate the agent's problem as a singular stochastic control problem. Shah (2005), Pommeret and Prieur (2013), Tsujimura (2014), and De Angelis and Ferrari (2018) also apply singular stochastic control to investigating environmental policies. See, for example, Karatzas (1983), Alvarez (2001), Yang and Liu (2004), and Pham (2006) for more details on singular stochastic control problems. Next, we formulate the agent problem as a stochastic impulse control problem in Case 2. Ferrari and Koch (2019) also examine a pollution control problem by applying impulse control. See, for instance, Eastham and Hastings (1988), Cadenillas and Zapatero (1999), Ohnishi and Tsujimura (2006) Alvarez and Lempa (2008), and Stokey (2009) for more details on stochastic impulse control problems. We then derive each optimal PRP as characterized by the thresholds required to invest in the projects. We derive the threshold of Case 1 explicitly, whereas we derive the threshold for Case 2 numerically. In addition, we employ comparative static analysis to investigate the effect of changes in the parameter values on the thresholds. We first find that the thresholds for both cases increase in the discount rate, the uncertainty parameter, and both fixed and proportional cost parameters. Second, the thresholds for both cases decrease with the drift parameter. Finally, we find that the impact of the damage parameter in Case 1 depends on the level of proportional cost. The results obtained offer useful guidance for the implementation of environmental policy.

The remainder of the paper is organized as follows. Section 2 describes the setup of the agent's problem. Section 3 examines Case 1 in which the PRP incurs proportional cost. Section 4 investigates Case 2 in which the PRP incurs fixed and proportional costs. We present the numerical analysis in Section 5. Section 6 concludes the paper.

2 Setup

Assume that an agent suffers from a pollutant emitted through an economic activity. The agent considers investing in a pollutant reduction project in order to mitigate damage. Let Y_t be the stock of the pollutant at time $t \geq 0$. We assume that when the agent does not reduce the pollutant, the following stochastic differential equation governs its dynamics:

$$\mathrm{d}Y_t = \mu Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}W_t, \quad Y_{0^-} = y, \tag{1}$$

where $\mu > 0$ is the expected growth rate of the pollutant stock including the rate of decay, $\sigma > 0$ represents the intensity of the uncertainty (or volatility), and W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$. We assume damage is dependent only on the magnitude of the pollutant stock and is continuously increasing in the pollutant stock. We justify this in that many pollution problems (such as climate change and acid rain) arise from the stock of pollutants. See, for example, Perman et al. (2003). The damage function $D(y)$ is assumed to be strictly convex and specified as a form of:

$$D(y) = ay^b, \tag{2}$$

where $a > 0$ is a conversion factor and $b > 1$ is the damage elasticity of the pollutant stock[3)]. The conversion factor a converts the magnitude of the pollutant stock to a monetary value. Further, we assume the damage function D satisfies:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} D(Y_t)\mathrm{d}t\right] < \infty, \tag{3}$$

where $r > 0$ is the discount rate. Let z_t be the cumulative amount of pollutant reduction until time t and be right continuous with left limits. For any $z \in BV$, we obtain the Lebesgue decomposition (Yong and Zhou, 1999, p.87):

$$z_t = z_t^{ac} + z_t^{sc} + z_t^{jp}, \quad t \in [0, \infty), \tag{4}$$

where BV is the space of bounded variational functions, z^{ac} is the absolutely continuous part of z, z^{sc} is the singularly continuous part of z, and z^{jp} is the pure jump part of z. The pure jump part of z is defined by $z_t^{jp} := \sum_{0 \leq s < t} \Delta z_s$, where $\Delta z_t := z_t - z_{t-}$. The continuous part of

z, z^c is defined by $z_t^c := z_t - z_t^{jp}$, so that $z_t^c = z_t^{ac}$. In the current analysis, the absolutely continuous part of z is assumed to be equivalently zero, $z^{ac} \equiv 0$, in order to focus on the two types of pollution reduction policies, namely, Case 1 and Case 2[4)]. The process of the pollutant stock then becomes:

$$Y_t = y + \int_0^t \mu Y_s \mathrm{d}s + \int_0^t \sigma Y_s \mathrm{d}W_s - z_t. \tag{5}$$

The agent's expected total discounted cost J is given by:

$$J(y; z) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} D(Y_t)\mathrm{d}t + \int_0^\infty \mathrm{e}^{-rt} K_{sc}(t)\mathrm{d}z_t^{sc} + \sum_{t\geq 0} \mathrm{e}^{-rt} K_{jp}(t, \Delta z_t) 1_{\{\Delta z_t \neq 0\}}\right], \tag{6}$$

where K_{sc} and K_{jp} denote reduction cost functions. Then, the agent's problem is to choose z to minimize the expected total cost J.

The next section is devoted to investigating the case in which the PRP incurs the proportional cost. Section 4 examines the case in which the PRP incurs the fixed and proportional costs. In this case, the singularly continuous part of z is equivalently zero, $z^{sc} \equiv 0$.

3 Pollution Reduction Policy with Proportional Cost

This section investigates the PRP of Case 1, that is, when the agent invests in the pollutant reduction project, it incurs only the proportional cost. For simplicity, we assume the proportional cost is constant. Then, the reduction cost functions K_{sc} and K_{jp} are the same form without the fixed cost. Then, replace these functions by the proportional cost, $k_p > 0$.

In this case, the cumulative amount of pollutant reduction until time t is $\zeta_t := z_t = z_t^{sc} + z_t^{jp}$. Then, the dynamics of pollutant stock (1) goes to:

$$\mathrm{d}Y_t = \mu Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}W_t - \mathrm{d}\zeta_t, \quad Y_{0^-} = y. \tag{7}$$

$\zeta := \{\zeta_t\}_{t\geq 0}$ is assumed to be nonnegative, nondecreasing, right-continuous with a left-hand limits $\mathcal{F}_t$-adapted process with $\zeta_{0^-} = 0$. Furthermore, we assume that:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}\zeta_t\right] < \infty. \tag{8}$$

The agent's expected total discounted cost J_{ssc} is then given by:

$$J_{ssc}(y; \zeta) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} D(Y_t)\mathrm{d}t + k_p \int_0^\infty \mathrm{e}^{-rt}\mathrm{d}\zeta_t\right], \tag{9}$$

Therefore, the agent's problem is to choose ζ so as to minimize J_{ssc}:

$$V_{ssc}(y) = \inf_{\zeta \in \ z} J_{ssc}(y; \zeta) = J_{ssc}(y; \zeta^*), \tag{10}$$

where V_{ssc} is the value function, $\mathcal{Z}$ is the set of admissible PRPs, and ζ^* is the optimal PRP. We formulate the agent's problem (10) as a singular stochastic problem.

From the formulation of the agent's problem (10), we naturally surmise that under an optimal PRP, the agent reduces the pollutant whenever the pollutant stock reaches some threshold $\overline{y}$. To verify this conjecture, we solve the agent's problem (10) using variational inequalities.

The variational inequalities of the agent's problem (10) are as follows Øksendal, 2003).

Definition 3.1 (Variational Inequalities).

The following relations are referred to as the variational inequalities for the agent's problem (10):

$$\mathcal{L}V_{ssc}(y) + D(y) \geq 0, \tag{11}$$

$$V'_{ssc}(y) \leq k_p, \tag{12}$$

$$[\mathcal{L}V_{ssc}(y) + D(y)][k_p - V'(y)] = 0, \tag{13}$$

where $\mathcal{L}$ *is the operator defined by:*

$$\mathcal{L} := \frac{1}{2}\sigma^2 y^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2} + \mu y \frac{\mathrm{d}}{\mathrm{d}y} - r. \tag{14}$$

Let H_{ssc} be the continuation region given by:

$$H_{ssc} = \{y; V'(y) < k_p\}. \tag{15}$$

The following lemma is the well-known Skorohod Lemma. A proof is available in Rogers and Williams (2000, pp.117-118).

Lemma 3.1.

For any $y > 0$ *and given a boundary* $\overline{y} > 0$, *there exist a unique cadlag-adapted process* $Y^* = \{Y_t^*\}_{t \geq 0}$ *and a nondecreasing process* ζ^* *satisfying the following Skorohod problem:*

$$\mathrm{d}Y_t^* = \mu Y_t^* \mathrm{d}t + \sigma Y_t^* \mathrm{d}W_t - \mathrm{d}\zeta_t, \;\; Y_{0-}^* = y, \;\; t \geq 0, \tag{16}$$

$$Y_t^* \in (0, \overline{y}] \;\; a.e., \;\; t \geq 0, \tag{17}$$

$$\int_0^t \mathbf{1}_{\{Y_s^* < \overline{y}\}} \mathrm{d}\zeta_s^* = 0. \tag{18}$$

Furthermore, if $y \leq \overline{y}$, *then* ζ^* *is continuous. If* $y > \overline{y}$, *then* $\zeta_0^* = y - \overline{y}$ *and* $Y_0^* = \overline{y}$.

The Skorohod Lemma implies that Y^* is a reflected diffusion at the boundary $\overline{y}$ and ζ^* is the local time of Y^* at $\overline{y}$. The condition (18) means that ζ^* increases only when Y^* reaches $\overline{y}$.

Then, the continuation region H_{ssc} is replaced by:

$$H_{ssc} = \{y; y < \overline{y}\}. \tag{19}$$

Let $\phi_{ssc} \in C^2$ be a function and $\tau(< \infty)$ be a stopping time. From the Ito formula for cadlag semimartingales we have:

$$\begin{aligned} e^{-r\tau}\phi_{ssc}(Y_\tau) = \phi_{ssc}(y) + \int_0^\tau e^{-r\tau}\mathcal{L}\phi_{ssc}(Y_t)dt + \int_0^\tau e^{-r\tau}\sigma Y_t \phi'_{ssc}(Y_t)dW_t \\ \int_0^\tau e^{-r\tau}\phi'_{ssc}(Y_t)dz_t^c + \sum_{0\le t\le\tau} e^{-r\tau}[\phi_{ssc}(Y_t) - \phi_{ssc}(Y_{t^-})]. \end{aligned} \tag{20}$$

Note that it follows from $z^{ac} \equiv 0$ that $z_t^c = z^{sc}$.

We are now in a position to prove that a solution to the variational inequalities is optimal. The following is the well-known verification theorem. We prove the theorem by following Pham (2006, Proposition 1.3.1) and Yang and Liu (2004, Theorem 1) in Appendix A.

Theorem 3.1

(I) Let ϕ_{ssc} be a solution of the variational inequalities satisfying the following:

$$\lim_{t\to\infty} e^{-rt}\phi_{ssc}(Y_t) = 0. \tag{21}$$

Then, we obtain:

$$\phi_{ssc}(y) \le V_{ssc}(y), \quad y > 0. \tag{22}$$

(II) ϕ_{ssc} also satisfies the following:

$$\mathcal{L}\phi_{ssc}(y) + D(y) = 0, \quad y < \overline{y}, \tag{23}$$

$$\phi_{ssc}(y) = k_p(y - \overline{y}) + c, \quad y \ge \overline{y}, \tag{24}$$

where c is constant. Then, there exists an optimal policy $\zeta^* \in \mathcal{Z}$ such that:

$$\phi_{ssc}(y) = V_{ssc}(y). \tag{25}$$

That is, ϕ_{ssc} is the value function and ζ^* is the corresponding optimal policy.

Proof. See Appendix A. □

For $y < \overline{y}$, the variational inequalities (11)-(13) lead to the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 y^2 \phi''_{ssc}(y) + \mu y \phi'_{ssc}(y) - r\phi_{ssc}(y) + D(y) = 0. \tag{26}$$

The general solution of the ordinary differential equation (26) with $D(y) = 0$ is given by:

$$\phi_{ssc}(y) = A_1 y^{\beta_1} + A_2 y^{\beta_2}, \quad y < \overline{y}, \tag{27}$$

where A_1 and A_2 are constants to be determined. The solutions to the following characteristic equation are β_1 and β_2:

$$\Gamma(\beta) := \frac{1}{2}\sigma^2\beta(\beta-1)+\mu\beta-r=0, \tag{28}$$

and are calculated with:

$$\beta_1=\frac{1}{2}-\frac{\mu}{\sigma^2}+\left[\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}\right]^{\frac{1}{2}}>1,\ \ \beta_2=\frac{1}{2}-\frac{\mu}{\sigma^2}-\left[\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}\right]^{\frac{1}{2}}<0, \tag{29}$$

where the inequalities of (29) hold with $r>\mu$ which is derived by assumption (3). On the other hand, to find a particular solution of (26), we seek a function of the form $\phi_p(y)=\lambda y^b$. As $\mathcal{L}\phi_p(y)+D(y)=0$, we have:

$$\lambda=\frac{a}{\rho}, \tag{30}$$

where $\rho := r-\mu b-(1/2)b(b-1)\sigma^2$. It follows from (3) that we have $\rho>0$. The general solution of (26) is:

$$\phi_{ssc}(y)=A_1y^{\beta_1}+A_2y^{\beta_2}+\frac{ay^b}{\rho},\ y<\overline{y}. \tag{31}$$

Given $\beta_2<0$, the boundary condition $\lim_{y\to 0}\phi_{ssc}(y)=0$ yields $A_2=0$. Then, the general solution to (26) is:

$$\phi_{ssc}(y)=A_1y^{\beta_1}+\frac{ay^b}{\rho},\ \ y<\overline{y}. \tag{32}$$

The second term on the right-hand side of (32) represents the expected discounted present value of damage when the agent will not reduce the pollutant forever:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}D(Y_t)\mathrm{d}t\right]=\frac{ay^b}{\rho}. \tag{33}$$

From the definition of the agent's problem, we have:

$$\phi_{ssc}(y)<\frac{ay^b}{\rho}. \tag{34}$$

Then, we obtain $A_1<0$.

Let ϕ_{ssc} be redefined as a candidate function for the value function given by:

$$\phi_{ssc}(y)=\begin{cases}\psi(y):=A_1y^{\beta_1}+\dfrac{ay^b}{\rho}, & y<\overline{y},\\ \psi(\overline{y})+k_p(y-\overline{y}), & y\geq\overline{y}.\end{cases} \tag{35}$$

Two unknowns, A_1 and $\overline{y}$, are determined by the following simultaneous equations:

$$\psi'(\overline{y})=k_p, \tag{36}$$

$$\psi''(\overline{y})=0. \tag{37}$$

The condition (36) is the smooth-pasting condition and (37) is the super contact condition (see

Dumas (1991) for details). From (36) and (37) we obtain:

$$A_1 = \frac{k_p(b-1)\overline{y}^{1-\beta_1}}{\beta_1(b-\beta_1)}, \tag{38}$$

$$\overline{y} = \left[\frac{\rho(\beta_1-1)k_p}{ab(\beta_1-b)}\right]^{\frac{1}{b-1}}. \tag{39}$$

Note that from $A_1 < 0$, (38), and (39), the parameter b must satisfy:

$$1 < b < \beta_1. \tag{40}$$

We can explore characteristics of the PRP in Case 1 by investigating the impact of varying the threshold parameter $\overline{y}$. We conduct a comparative static analysis in Section 5 so that we can compare the results of both cases.

4 Pollution Reduction Policy with Fixed and Proportional Costs

In this section, we investigate the case in which implementing the PRP incurs a fixed cost and a cost proportional to the reduction.

Let k_f be the fixed cost and a constant. Given the presence of a fixed cost, it is not optimal to reduce the pollutant continuously. Then, the cumulative amount of pollutant reduction until time t is $z_t = z_t^{jp}$ with $z_t^{sc} \equiv 0$. This is verified by the fact that the reduction cost function satisfies subadditivity with respect to the reduction amount Δz_t:

$$K_{jp}(\Delta z_t + \Delta z_t') \leq K_{jp}(\Delta z_t) + K_{jp}(\Delta z_t'), \tag{41}$$

where the reduction cost function K_{jp} is given by:

$$K_{jp}(\Delta z_t) = k_f + k_p \Delta z_t. \tag{42}$$

Let ξ_i be the i th amount of pollutant reduction and τ_i be its time, where $0^- = \tau_{0^-} < \tau_1 < \cdots < \tau_i < \cdots$. Then, $\xi_i = z_{\tau_i} - z_{\tau_i -}$ and $1_{\{\tau_i\}} = 1_{\{\Delta z_t \neq 0\}}$. We then replace the reduction cost function by:

$$K_{jp}(\xi) = k_f + k_p \xi. \tag{43}$$

An agent's PRP v is then defined as the following double sequences:

$$v = \{(\tau_i, \xi_i)\}_{i \geq 0}. \tag{44}$$

For all $i \geq 0$, the dynamics of the pollutant stock (1) changes to:

$$\begin{cases} dY_t = \mu Y_t dt + \sigma Y_t dW_t, & \tau_i \leq t < \tau_{i+1} < \infty, \\ Y_{\tau_i} = Y_{\tau_i -} - \xi_i, \\ Y_{0-} = y. \end{cases} \tag{45}$$

We assume that τ_i satisfies:

$$\mathbb{P}\left\{\lim_{i\to\infty}\tau_i \le \tilde{T}\right\} = 0, \tag{46}$$

where $\tilde{T}$ is a terminal time. The condition (46) implies that pollutant reduction will only occur finitely before $\tilde{T}$. Then the agent's expected total discounted cost function J_{im} is given by:

$$J_{im}(y;v) = \mathbb{E}\left[\int_0^\infty e^{-rt}D(Y_t)\mathrm{d}t + \sum_{i=0}^\infty e^{-r\tau_i}K_{jp}(\xi_i)\mathbf{1}_{\{\tau_i<\infty\}}\right]. \tag{47}$$

Therefore, the agent's problem is to choose v so as to minimize J_{im}:

$$V_{im}(y) = \inf_{v\in\mathcal{V}} J_{im}(y;v) = J_{im}(y;v^*), \tag{48}$$

where V_{im} is the value function, $\mathcal{V}$ is the set of admissible PRPs and v^* is the optimal PRP. We formulate the agent's problem (48) as a stochastic impulse control problem.

From the formulation of the agent's problem (48), we naturally surmise that an optimal PRP is in the following form specified by two critical pollutant levels: namely, whenever the pollutant stock reaches some level $\tilde{y}$, the agent reduces the pollutant, so that it instantaneously reduces to another pollutant level $\hat{y}$. To verify this conjecture, we solve the agent's problem (48) using quasi-variational inequalities.

Let $\mathcal{M}$ be the pollutant reduction operator defined by:

$$\mathcal{M}V_{im}(y) = \inf_{\xi\in[0,y)}\{(k_f + k_p\xi) + V_{im}(y-\xi)\}. \tag{49}$$

Then, the quasi-variational inequalities (QVI) of the agent's problem (48) are as follows (Bensoussan and Lions, 1984):

Definition 4.1 (QVI)

The following relations are the QVI of the agent's problem (48):

$$\mathcal{L}V_{im}(y) + D(y) \ge 0, \tag{50}$$

$$V_{im}(y) \le \mathcal{M}V_{im}(y), \tag{51}$$

$$[\mathcal{L}V_{im}(y) + D(y)][\mathcal{M}V_{im}(y) - V_{im}(y)] = 0. \tag{52}$$

From the solution to the QVI, it is possible to construct the following impulse control.

Definition 4.2 (QVI-policy)

Let ϕ be a solution to the QVI. Then the following PRP $\hat{v} = \{(\hat{\tau}_i, \hat{\xi}_i)\}_{i\ge 0}$ is the QVI policy:

$$(\hat{\tau}_0, \hat{\xi}_0) = (0, 0), \tag{53}$$

$$\hat{\tau}_i = \inf\{t > \hat{\tau}_{i-1}; Y_t \notin H_{im}\}, \tag{54}$$

$$\hat{\xi}_i = \arg\min\{\phi(Y_{\hat{\tau}_i} - \nu) + (k_f + k_p \nu); \nu \geq 0, Y_{\hat{\tau}_i} - \nu > 0\}. \tag{55}$$

Here, H_{im} is the continuation region defined by:

$$H_{im} = \{y; \phi(y) < \mathcal{M}\phi(y)\}. \tag{56}$$

Now we are in a position to prove that the QVI policy is the optimal PRP. The following is the well-known verification theorem. We mainly refer to Brekke and Øksendal (1998, Theorem 3.1) and Cadenillas and Zapatero (1999, Theorem 3.1).

Theorem 4.1

(I) Let ϕ be a solution of the QVI. Suppose that ϕ is a C^1 function for $y > 0$ and is a C^2 function for y in $(0, \infty) - \mathcal{N}$, where $\mathcal{N}$ is a finite subset of $(0, \infty)$. Suppose that there exists $U \in (0, \infty)$ such that ϕ is linear in $y \in (U, \infty)$ and satisfies:

$$\lim_{t \to \infty} \mathrm{e}^{-rt} \phi(Y_t) = 0. \tag{57}$$

Then, for all $y > 0$ we obtain:

$$\phi(y) \leq V_{im}(y). \tag{58}$$

(II) From the QVI, we have:

$$\mathcal{L}\phi(y) + D(y) = 0, \quad y \in H_{im}, \tag{59}$$

Suppose that QVI policy $\hat{v}$ is admissible. Then we obtain:

$$\phi(y) = V_{im}(y). \tag{60}$$

That is, ϕ is the value function and $\hat{v}$ is the corresponding optimal policy.

Proof. See Appendix B. □

From the above conjecture, the continuation region H_{im} is given by:

$$H_{im} = \{y; y < \tilde{y}\}. \tag{61}$$

Then, an optimal PRP $v^* = (\tau^*, \xi^*)$ is characterized by $\tilde{y}$ and $\hat{y}$ with $0 < \hat{y} < \tilde{y} < \infty$ such that:

$$\tau_i^* = \inf\{t > \tau_{i-1}^*; X_{t^-} \notin H_{im}\}, \tag{62}$$

$$\xi_i^* = X_{\tau_i^{*-}} - X_{\tau_i^*} = \tilde{y} - \hat{y}. \tag{63}$$

Let $\phi_{im} \in C^2$ be a function. For $y < \tilde{y}$, QVI (50)-(52) lead to the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 y^2 \phi_{im}''(y) + \mu y \phi_{im}'(y) - r\phi_{im}(y) + ay^b = 0. \tag{64}$$

As in Section 3, the solution to (64) is:

$$\phi_{im}(y) = B_1 y^{\beta_1} + \frac{ay^b}{\rho}, \tag{65}$$

where B_1 is a constant to be determined and β_1 is derived by (29). The value of B_1 is negative along with A_1. Let ϕ_{im} be defined as a candidate function of the value function given by:

$$\phi_{im}(y) = \begin{cases} \varphi(y) := B_1 y^{\beta_1} + \dfrac{ay^b}{\rho}, & y < \tilde{y} \\ \varphi(\hat{y}) + k_f + k_p(y - \hat{y}), & y \geq \tilde{y} \end{cases} \tag{66}$$

Three unknowns B_1, $\tilde{y}$, and $\hat{y}$ are determined by the following simultaneous equations. The first equation is:

$$\varphi(\tilde{y}) = k_f + k_p(\tilde{y} - \hat{y}) + \varphi(\hat{y}). \tag{67}$$

The second equation is:

$$\begin{aligned} \varphi'(\tilde{y}) &= \lim_{y \downarrow \tilde{y}} \varphi'(y) \\ &= \lim_{y \downarrow \tilde{y}} \frac{\mathrm{d}}{\mathrm{dy}} [k_f + k_p(y - \hat{y}) + \varphi(\hat{y})] \\ &= k_p. \end{aligned} \tag{68}$$

From (62) and (63), J_{im} is minimized at $\xi^* = \tilde{y} - \hat{y}$:

$$\begin{aligned} \varphi(\tilde{y}) &= k_f + k_p(\tilde{y} - \hat{y}) + \varphi(\hat{y}) \\ &= \max_{q \in [0, \tilde{y}]} [k_f + k_p(\tilde{y} - q) + \varphi(q)]. \end{aligned} \tag{69}$$

The final equation is:

$$\varphi'(\hat{y}) = k_p.$$

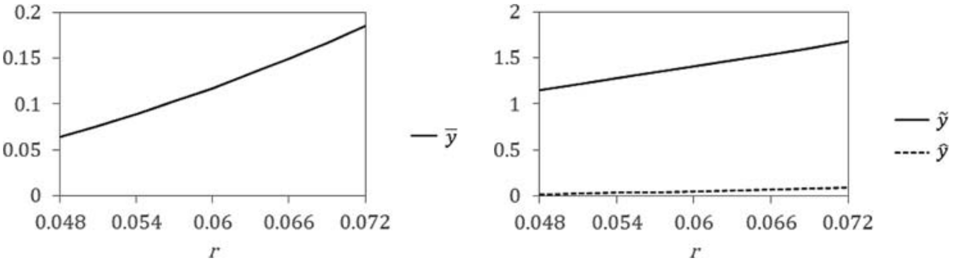
Unfortunately, as we are unable to derive these unknowns analytically, in the following section we numerically calculate their values. We then compare the results of the comparative static analysis for both cases.

5 Numerical Analysis

In this section, we numerically examine both of the optimal PRPs in order to explore their characteristics. All of the parameter values are identical in each case so that we can best compare the impact of varying the parameter values at the thresholds. The basic parameter values are set out as follows: $r = 0.06$, $\mu = 0.015$, $\sigma = 0.15$, $a = 0.01$, $b = 1.5$, $k_p = 0.1$, and $k_f = 0.1$. Then, we obtain $A_1 = -0.4189$, $B_1 = -0.1608$, $\bar{y} = 0.1177$, $\tilde{y} = 1.4102$, and $\hat{y} = 0.0453$.

We provide the results of the comparative static analysis of the thresholds in Figures 1-8. Figure 1 shows that the continuation regions H_{ssc} and H_{im}, and the amount of pollutant reduction $\tilde{y} - \hat{y}$ are increasing in the discount rate, r. The higher the discount rate implies that the smaller present value of the damage. The agent then postpones implementing the PRPs. Once the PRP is implemented in Case 2, the amount of pollutant reduction increases.

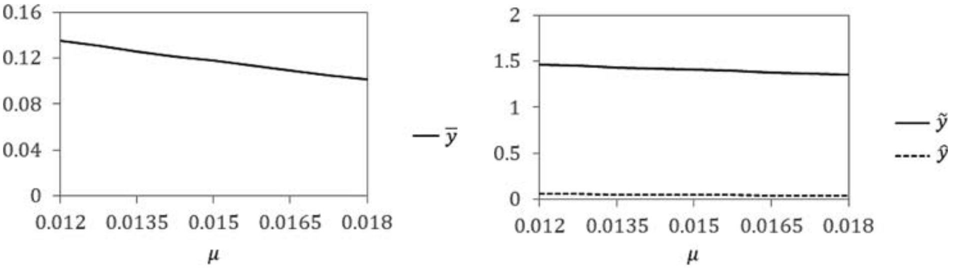
Figure 1 Comparative statics of the thresholds with respect to r.



(a) Case 1 (b) Case 2

Figure 2 illustrates that the continuation regions H_{ssc} and H_{im}, and the amount of reduction are decreasing in the expected growth rate of the pollutant stock, μ. Put differently, the higher the expected growth rate of the pollutant stock, the larger the present value of the damage. Accordingly, the agent hastens the implementation of the PRPs.

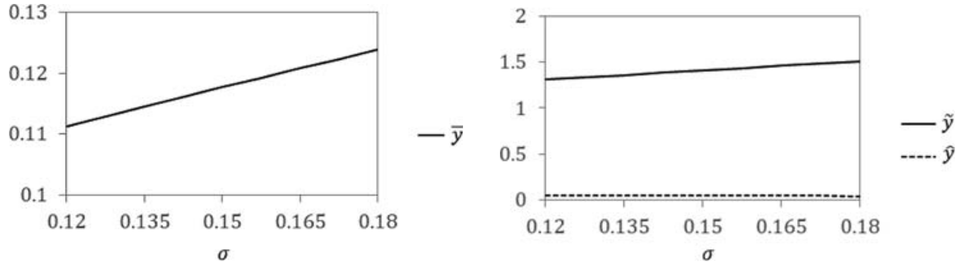
Figure 2 Comparative statics of the thresholds with respect to μ.



(a) Case 1 (b) Case 2

Figure 3 depicts that the continuation regions H_{ssc} and H_{im}, and the amount of reduction are increasing in the volatility of the pollutant stock, σ. These results imply that the incentive to wait for new information regarding the damage becomes stronger as the uncertainty in the dynamics of pollutant stock increases. Consequently, the agent postpones implementation of the PRPs.

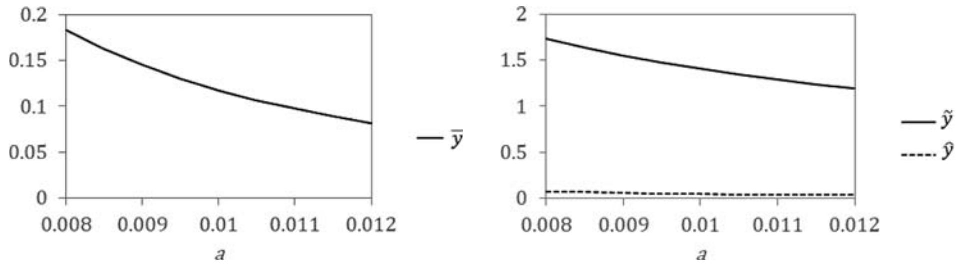
Figure 3 Comparative statics of the thresholds with respect to σ.



(a) Case 1 (b) Case 2

Figure 4 shows that the continuation regions H_{ssc} and H_{im}, and the amount of reduction are decreasing in the damage parameter a. This implies that the higher the damage parameter a, the larger the present value of the damage. Accordingly, the agent hastens the implementation of the PRPs.

Figure 4 Comparative statics of the thresholds with respect to a.

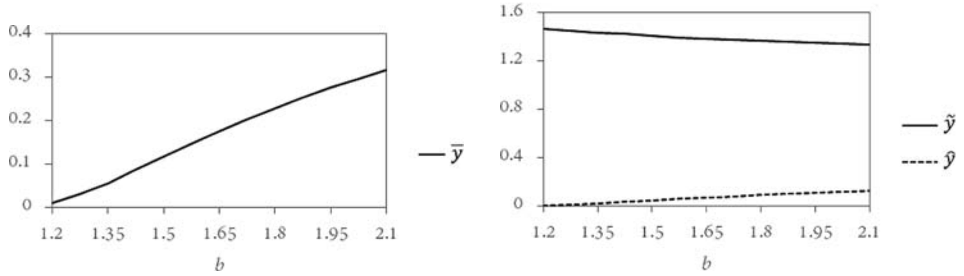


(a) Case 1 (b) Case 2

Figure 5 illustrates that the continuation region H_{ssc} is increasing in the damage elasticity of the pollutant stock b, while the continuation region H_{im} is decreasing in b. In general, the higher the damage elasticity of the pollutant stock, the larger the present value of the damage. This implies the result of H_{im}. The unexpected result for H_{ssc} comes from the setting of parameter values. The sensitivity of H_{ssc} with respect to b is depended on the value of the proportional cost k_p. See Appendix C. for details. In our base parameter values, if k_p is higher than 0.236796 approximately, the derivative of $\bar{y}$ with respect to b is negative. This means that the continuation region H_{ssc} is decreasing in b. Figure 6 illustrates that the sensitivity of $\bar{y}$ with several values of k_p.

The sensitivity of the thresholds with respect to k_p itself is illustrated in Figure 7. It shows that the continuation regions H_{ssc} and H_{im}, and the amount of reduction are increasing in the proportional cost, k_p. Accordingly, the agent again postpones implementation of the PRPs.

Figure 5 Comparative statics of the thresholds with respect to b.



(a) Case 1 (b) Case 2

Figure 6 Comparative statics of $\bar{y}$ with respect to b with several k_p values from 0.05 to 0.3.

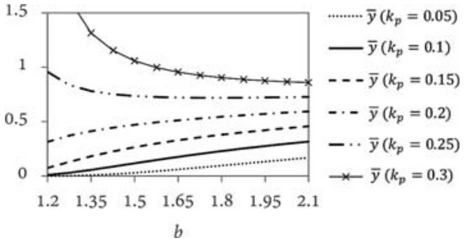
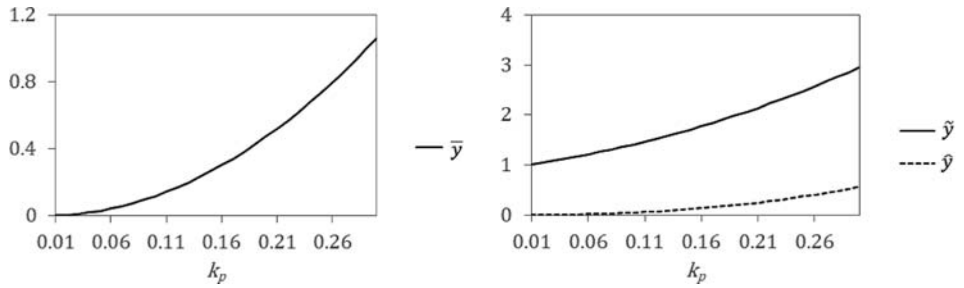


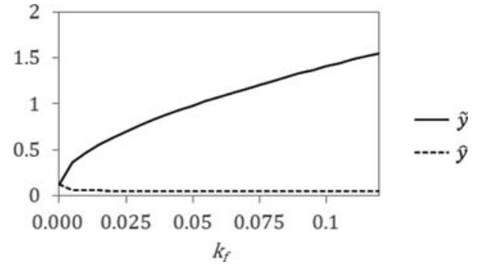
Figure 7 Comparative statics of the thresholds with respect to k_p.



(a) Case 1 (b) Case 2

Figure 8 shows that the continuation region H_{im}, and the amount of reduction are increasing in the fixed cost, k_f. Accordingly, the agent postpones implementation of the PRPs. When the fixed cost k_f goes to 0, the thresholds $\tilde{y}$ and $\hat{y}$ go to the threshold $\bar{y}$. The limit of the thresholds with the impulse control problem as the fixed cost k_f goes to 0 is equal to the threshold of the corresponding singular control problem as in Jeanblanc-Picqué and Shiryaev (1995).

Figure 8　Comparative statics of the thresholds with respect to k_f in Case 2.



6 Conclusion

In this paper, we examined PRPs under uncertainty. When the policy is implemented by the agent, it contains two types of reduction costs. We formulated the agent's problems as the singular stochastic control problem and the stochastic impulse control problem, respectively. We found optimal PRPs in both cases. In the paper, we also presented the results of a numerical analysis. Our main results are as follows. First, the thresholds for both PRPs increase in the discount rate, the uncertainty parameter (volatility), and the cost parameters. Second, the thresholds for both policies decrease with the drift parameter. Finally, we found there is a difference in the impact of the damage elasticity of the pollutant stock b on both policies.

To conclude the paper, we suggest a number of possible extensions for our model. To start with, we need to use this framework to undertake the examination of specific pollutants and/or pollutant reduction projects. Furthermore, we would consider the model uncertainty of pollutant stock dynamics. When we consider environmental policies, we often face long-term decision-making problems like climate change related policy issues. We need to incorporate the limits of our knowledge into the policy analysis. To this end, we would consider the environmental policy under model uncertainty (Funke and Paetz 2011; Tsujimura, 2015). We leave these important topics to future research.

Appendix A.

Proof of therem 3.1 (I) For $\{z_t\}_{t\geq 0} \in \mathcal{Z}$, let $\tau_n = \inf\{t \geq 0; Y_t \geq n\} \wedge n$, $n \in \mathbb{N}$ be the finite stopping time. We apply (20) between $t = 0$ and $t = \tau_n$ and take expectation. We obtain that:

$$\mathbb{E}[\mathrm{e}^{-r\tau_n}\phi_{ssc}(Y_{\tau_n})] = \phi_{ssc}(y) + \mathbb{E}\left[\int_0^{\tau_n} \mathrm{e}^{-rt}{\scriptstyle \mathcal{L}}\phi_{ssc}(Y_t)\mathbf{x}t\right] - \mathbb{E}\left[\int_0^{\tau_n} \mathrm{e}^{-rt}\phi'_{ssc}(Y_t)\mathrm{d}z_t^c\right]$$

$$\mathbb{E}\left[\sum_{0\le t\le\tau_n} \mathrm{e}^{-rt}[\phi_{ssc}(Y_t) - \phi_{ssc}(Y_t^-)]\right]. \tag{A.1}$$

As (12) and $Y_t - Y_{t^-} = -\Delta z_t$ indicate, the mean-value theorem implies that:

$$\phi_{ssc}(Y_t) - \phi_{ssc}(Y_{t^-}) = -\phi_{ssc}(\theta)\Delta z_t \ge -k_p\Delta z_t, \tag{A.2}$$

where $\theta \in (Y_{t^-}, Y_t)$. It follows from (11) and (12) that (A.1) is rewritten as:

$$\mathbb{E}[\mathrm{e}^{-r\tau_n}\phi_{ssc}(Y_{\tau_n})] \ge \phi_{ssc}(y) - \mathbb{E}\left[\int_0^{\tau_n} \mathrm{e}^{-rt}D(Y_t)\mathrm{d}t\right] - \mathbb{E}\left[\int_0^{\tau_n} \mathrm{e}^{-rt}k_p\,\mathrm{d}z_t^c\right]$$

$$-\mathbb{E}\left[\sum_{0\le t\le\tau_n} \mathrm{e}^{-rt}k_p\Delta z_t\right]. \tag{A.3}$$

Further, it follows from $z_t^c = z_t - \sum_{0\le s\le t}\Delta z_s$ and $\zeta_t := z_t$ that we have:

$$\mathbb{E}[\mathrm{e}^{-r\tau_n}\phi_{ssc}(Y_{\tau_n})] \ge \phi_{ssc}(y) - \mathbb{E}\left[\int_0^{\tau_n} \mathrm{e}^{-rt}D(Y_t)\mathrm{d}t + \int_0^{\tau_n} \mathrm{e}^{-rt}k_p\,\mathrm{d}\zeta_t\right]. \tag{A.4}$$

Taking $\lim_{n\to\infty}$ and using (21) and the dominated convergence theorem, we obtain that:

$$\phi_{ssc}(y) \le \mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-rt}D(Y_t)\mathrm{d}t + \int_0^{\infty} \mathrm{e}^{-rt}k_p\,\mathrm{d}\zeta_t\right] = J(y;\zeta). \tag{A.5}$$

From the arbitrariness of ζ, we have:

$$\phi_{ssc}(y) \le \inf_{\zeta\in\mathcal{Z}} J_{ssc}(y;\zeta) = V_{ssc}(y), \tag{A.6}$$

which completes the proof of (I).

(II) For $y < \overline{y}$, from Lemma 3.1, ζ^* is continuous for all $y < \overline{y}$ and increases only when $Y^* = \overline{y}$. Then, for $\zeta = \zeta^*$ (A.5) becomes equality:

$$\phi_{ssc}(y) = \mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-rt}D(Y_t)\mathrm{d}t + \int_0^{\infty} \mathrm{e}^{-rt}k_p\,\mathrm{d}\zeta_t\right] = J(y;\zeta^*) = V_{ssc}(y). \tag{A.7}$$

For $y \ge \overline{y}$, it follows from Lemma 3.1 that we have:

$$V_{ssc}(y) = k_p(y - \overline{y}) + V_{ssc}(\overline{y}). \tag{A.8}$$

From (A.7) we have $\phi_{ssc}(\overline{y}) = V_{ssc}(\overline{y})$. From the continuous property of $\phi_{ssc}(y)$, $\phi_{ssc}(\overline{y}) = c$. Thus, for all $y \ge \overline{y}$ we have:

$$V_{ssc}(y) = k_p(y - \overline{y}) + \phi_{ssc}(\overline{y}) = \phi_{ssc}(y). \tag{A.9}$$

This completes the proof of (II). □

Appendix B.

Proof of Theorem 4.1. (I) For all $t > 0$ and $i \in \mathbb{N}$ we obtain:

$$\mathrm{e}^{-r(t\wedge\tau_n)}\phi_{im}(Y_{(t\wedge\tau_n)}) = \phi_{im}(y) + \sum_{i=1}^{n}\left[\mathrm{e}^{-r(t\wedge\tau_i)}\phi_{im}(Y_{(t\wedge\tau_i)^-}) - \mathrm{e}^{-r(t\wedge\tau_{i-1})}\phi_{im}(Y_{(t\wedge\tau_{i-1})})\right]$$
$$+ \sum_{i=1}^{n}\left[\mathbf{1}_{\{\tau_n<t\}}\mathrm{e}^{-r\tau_i}(\phi_{im}(Y_{\tau_i}) - \phi_{im}(Y_{\tau_i^-}))\right], \tag{B.10}$$

where Y is a continuous semimartingale in $t \in [\tau_{i-1}, \tau_i)$ and ϕ_{im} is a C^2 function in $y \in (0,\infty) - \mathcal{N}$. Then, it follows from the Ito formula for a semimartingale that:

$$\mathrm{e}^{-r(t\wedge\tau_i)}\phi_{im}(Y_{(t\wedge\tau_i)^-}) = \mathrm{e}^{-r(t\wedge\tau_{i-1})}\phi_{im}(Y_{t\wedge\tau_{i-1}}) + \int_{t\wedge\tau_{i-1}}^{(t\wedge\tau_i)^-} \mathrm{e}^{-rs}\mathcal{L}\phi_{im}(Y_s)\mathrm{d}s$$
$$+ \int_{t\wedge\tau_{i-1}}^{(t\wedge\tau_i)^-} \mathrm{e}^{-rs}\sigma Y_s\phi'_{im}(Y_s)\mathrm{d}W_s. \tag{B.11}$$

Using (50), we can rewrite (B.11) as:

$$\mathrm{e}^{-r(t\wedge\tau_i)}\phi_{im}(Y_{(t\wedge\tau_i)^-}) \geq \mathrm{e}^{-r(t\wedge\tau_{i-1})}\phi_{im}(Y_{t\wedge\tau_{i-1}}) - \int_{t\wedge\tau_{i-1}}^{(t\wedge\tau_i)^-} \mathrm{e}^{-rs}D(Y_s)\mathrm{d}s$$
$$+ \int_{t\wedge\tau_{i-1}}^{(t\wedge\tau_i)^-} \mathrm{e}^{-rs}\sigma Y_s\phi'_{im}(Y_s)\mathrm{d}W_s. \tag{B.12}$$

Moreover, from (51) we have:

$$\mathrm{e}^{-r\tau_i}[\phi_{im}(Y_{(\tau_i)^-}) - \phi_{im}(Y_{\tau_i})] \leq \mathrm{e}^{-r\tau_i}K_{jp}(\xi_i). \tag{B.13}$$

Combining (B.10)-(B.13), and taking expectations, we obtain:

$$\mathbb{E}\left[\mathrm{e}^{-r(t\wedge\tau_i)}\phi_{im}(Y_{(t\wedge\tau_i)^-})\right] \geq \phi_{im}(y) - \mathbb{E}\left[\sum_{i=1}^{n}\left[\int_{t\wedge\tau_{i-1}}^{(t\wedge\tau_i)^-} \mathrm{e}^{-rs}D(Y_s)\mathrm{d}s + \mathbf{1}_{\{\tau_i<t\}}\mathrm{e}^{-r\tau_i}K_{jp}(\xi_i)\right]\right]. \tag{B.14}$$

Taking $\lim_{n\to\infty}$ and $\lim_{t\to\infty}$ and using (57) and the dominated convergence theorem, we obtain:

$$\phi_{im}(y) \leq \int_0^{\infty} \mathrm{e}^{-rs}D(Y_s)\mathrm{d}s + \sum_{i=1}^{\infty}\mathbf{1}_{\{\tau_i<\infty\}}\mathrm{e}^{-r\tau_i}K_{jp}(\xi_i) = J_{im}(y;v). \tag{B.15}$$

For the arbitrariness of v, we have:

$$\phi_{im}(y) \leq \inf_{v\in\mathcal{V}} J_{im}(y;v) = V_{im}(y). \tag{B.16}$$

(II) Assume that (60) holds and the QVI policy $\hat{v}$ is applied. Repeating the argument in part (I) for $v = \hat{v}$, we have:

$$\phi_{im}(y) = J_{im}(y;\hat{v}) = V_{im}(y). \tag{B.17}$$

Therefore, ϕ_{im} is equivalent to the value function and the QVI policy is optimal. □

Appendix C.

Let rewrite the the threshold $\overline{y}$ in Case 1 as:

$$\overline{y} = \left[\frac{h}{g}\right]^{f}, \tag{C.18}$$

where $h = h(r, \mu, \sigma, b, k_p) := \rho(\beta_1 - 1)k_p$, $g = g(r, \mu, \sigma, a, b) := ab(\beta_1 - b)$, and $f = f(b) := \dfrac{1}{b-1}$. Notice that $\rho := r - \mu b - \dfrac{1}{2}b(b-1)\sigma^2$ and $\beta_1 = \dfrac{1}{2} - \dfrac{\mu}{\sigma^2} + \left[\left(\dfrac{\mu}{\sigma^2} - \dfrac{1}{2}\right)^2 + \dfrac{2r}{\sigma^2}\right]^{1/2}$.

Then, the derivative of $\overline{y}$ with respect to the damage elasticity of the pollutant stock, b, is:

$$\overline{y}_b := \frac{\partial \overline{y}}{\partial b} = \left[\frac{h}{g}\right]^{f} \left\{ \log\left[\frac{h}{g}\right] f' + \frac{f(h_b g - h g_b)}{h g} \right\}, \tag{C.19}$$

where h_b and g_b are the partial derivatives of h and g with respect to b, respectively. The sign of $\overline{y}_b$ is dependent on the sign of the braces term. The braces term can be rewritten as:

$$-\log\left[\frac{\rho(\beta_1 - 1)k_p}{ab(\beta_1 - b)}\right]\frac{1}{b-1} - \left[\frac{\mu + \frac{\sigma^2}{2}(2b-1)}{r - (\mu - \frac{\sigma^2}{2})b - \frac{\sigma^2}{2}b^2} + \frac{\beta_1 - 2b}{b\beta_1 - b^2}\right]. \tag{C.20}$$

The sign of the first term is dependent on the proportional cost k_p:

$$\log\left[\frac{\rho(\beta_1 - 1)k_p}{ab(\beta_1 - b)}\right] \begin{cases} \geq 0, & k_p \geq \dfrac{\beta_1 - b}{\beta_1 - 1}\dfrac{ab}{\rho}, \\ < 0, & \text{otherwise}. \end{cases} \tag{C.21}$$

The second term is positive since $\beta_1 > b$. Therefore, if the proportional cost k_p is enough high such that the braces term (C.20) is negative, the partial derivative $\overline{y}_b$ is also negative. In our numerical analysis, if k_p is higher than 0.236796, $\overline{y}_b$ is negative.

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Notes

1）The value of flexibility is known as the quasi-option value. The relationship between the quasi-option and real option values is discussed in Fisher (2000), Mensink and Requate (2005).

2）Décamps et al. (2006), Siddiqui and Fleten (2010), and Heydari et al. (2012) investigate the case in which the agent can choose a project from among two projects.

3）Pindyck (2000) assumes that $b = 1$. Pindyck (2002), Saphores and Carr (2000), Saphores

(2004), and Lin et al. (2007) assume that $b = 2$.

４）Cadenillas and Zapatero (2000) investigate the exchange rate control problem using an absolutely continuous control and an impulse control, while Davis and Norman (1990) examine an investor's consumption and investment problem using an absolutely continuous control and a singular stochastic control. Conversely, Øksendal and Sulem (2002) examine the investor's problem using an absolutely continuous control and an impulse control. Liu (2004) extends these studies and examines the same problem using an absolutely continuous control, a singular stochastic control, and an impulse control. However, Liu (2004) does not examine the threshold property discussed in this paper.

References

Alvarez, L. H. (2001), Singular stochastic control, linear diffusions, and optimal stopping: a class of solvable problems, *SIAM Journal on Control and Optimization,* **39**(6), 1697-1710.

Alvarez, L. H. and J. Lempa (2008), On the optimal stochastic impulse control of linear diffusions. SIAM Journal on Control and Optimization, **47**(2), 703-732.

Arrow, K. J. and A.C. Fisher (1974), Environmental Preservation, Uncertainty, and Irreversibility, *Quarterly Journal of Economics,* **88**, 312-319.

Balikcioglu, M., P. L. Fackler, and R. S. Pindyck (2011), Solving optimal timing problems in environmental economics, *Resource and Energy Economics,* **33**(3), 761-768.

Bensoussan, A. and J. L. Lions (1984), *Impulse Control and Quasi-Variational Inequalities,* Gauthier-Villars, Paris.

Brekke, K. A. and B. Øksendal (1998), A Verification Theorem for Combined Stochastic Control and Impulse Control, in: Decreusefond, L. J. Gjerde, B. Øksendal and A. S. Üstünel. (eds.), *Stochastic Analysis and Related Topics,* Vol.6. Bikhäuser, Basel, 211-220.

Cadenillas, A. and F. Zapatero (1999), Optimal Central Bank Intervention in the Foreign Exchange Market, *Journal of Economic Theory,* **87**, 218-242.

Cadenillas, A. and F. Zapatero, (2000), Classical and Impulse Stochastic Control of the Exchange Rate Using Interest Rates and Reserves, *Mathematical Finance,* **10**, 141-156.

Conrad, J. M. (1997), Global Warming: When to Bite the Bullet, *Land Economics,* **73**, 164-173.

Davis, M. H. A. and A. R. Norman (1990), Portfolio Selection with Transaction Costs, *Mathematics of Operations Research,* **15**, 676-713.

De Angelis, T. and Ferrari, G. (2018), Stochastic nonzero-sum games: a new connection between singular control and optimal stopping, *Advances in Applied Probability,* **50**(2), 347-372.

Décamps, J.-P., T. Mariotti and S. Villeneuve (2006), Irreversible Investment in Alternative Projects, *Economic Theory,* **28**, 425-448.

Dumas, B. (1991), Super Contact and Related Optimality Conditions, *Journal of Economic Dynamics and Control,* **15**, 675-685.

Eastham, J. F. and K. J. Hastings (1988), Optimal impulse control of portfolios, *Mathematics of Operations Research,* **13**(4), 588-605.

Ferrari, G. and T. Koch (2019). On a strategic model of pollution control. *Annals of Operations Research,* **275**(2), 297-319.

Fisher, A. C. (2000), Investment under Uncertainty and Option Value in Environmental Economics, *Resource and Energy Economics,* **22**, 197-204.

Funke, M. and Paetz, M. (2011), Environmental policy under model uncertainty: a robust optimal control approach, *Climatic Change,* **107**(3-4), 225-239.

Henry, C. (1974), Option Value in the Economics of Irreplaceable Assets, *Review of Economic Studies,* **41**, 89-104.

Heydari, S., N. Ovenden and A. Siddiqui (2012), Real Options Analysis of Investment in Carbon Capture and Sequestration Technology, *Computational Management Science,* **9**, 109-138.

IPCC (2005), *Carbon Dioxide Capture and Storage: Special Report of the Intergovernmental Panel on Climate Change,* Cambridge University Press.

Jeanblanc-Picqué, M. and A. N. Shiryaev (1995), Optimization of the Flow of Dividends, *Russian Mathematical Surveys,* **50**, 257-277.

Karatzas, I. (1983), A class of singular stochastic control problems. *Advances in Applied Probability,* **15**(2), 225-254.

Lin, T. T. and S.-L. Huang (2010), An Entry and Exit Model on the Energy-Saving Investment Strategy with Real Options, *Energy Policy,* **38**, 794-802.

Lin, T. T., C.-C. Ko and H.-N. Yeh (2007), Applying Real Options in Investment Decisions Relating to Environmental Pollution, *Energy Policy,* **35**, 2426-2432.

Liu, H. (2004), Optimal Consumption and Investment with Transaction Costs and Multiple Risky Assets, *Journal of Finance,* **59**, 289-338.

Leung, D. Y., G. Caramanna, and M. M. Maroto-Valer (2014), An overview of current status of carbon dioxide capture and storage technologies, *Renewable and Sustainable Energy Reviews,* **39**, 426-443.

Mensink, P. and T. Requate (2005), The Dixit-Pindyck and the Arrow-Fisher-Hanemann-Henry Option Values Are Not Equivalent: A Note on Fisher (2000), *Resource and Energy Economics,* **27**, 83-88.

Ohnishi, M. and M. Tsujimura (2006), An Impulse Control of a Geometric Brownian Motion with Quadratic Costs, *European Journal of Operational Research,* **168**, 311-321.

Øksendal, B. (2003), *Stochastic differential equations: an introduction with applications,* Springer-Verlag, Berlin Heidelberg.

Øksendal, B. and A. Sulem (2002), Optimal Consumption and Portfolio with Both Fixed and Proportional Transaction Costs, *SIAM Journal on Control and Optimization,* **40**, 1765-1790.

Perman, R., Y. Ma, J. McGilvray and M. Common (2003), *Natural Resource and Environmental*

Economics 3rd ed., Pearson Education, Harlow.

Pham, H. (2006), Explicit Solution to an Irreversible Investment Model with a Stochastic Production Capacity, in: Kabanov, Y., R. Lipster and J. Stoyanov (eds.), *From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift,* Springer-Verlag, Berlin, 547-566.

Pindyck, R. S. (2000), Irreversibilities and the Timing of Environmental Policy, *Resource and Energy Economics,* **22**, 233-259.

Pindyck, R. S. (2002), Optimal Timing Problems in Environmental Economics, *Journal of Economic Dynamics and Control,* **26**, 1677-1697.

Pindyck, R. S. (2007), Uncertainty in Environmental Economics, *Review of Environmental Economics and Policy,* **1**(1), 45-65.

Pommeret, A. and Prieur, F. (2013), Double irreversibility and environmental policy timing, *Journal of Public Economic Theory,* **15**(2), 273-291.

Rogers, L. C. G. and D. Williams (2000), *Diffusions, Markov Processes and Martingale: Volume 2: Ito Calculus,* 2nd ed., Cambridge University Press, Cambridge.

Saphores, J. D. M. (2004), Environmental Uncertainty and the Timing of Environmental Policy, *Natural Resource Modeling,* **17**, 163-190.

Saphores, J. D. M. and P. Carr (2000), Real Options and the Timing of Implementation of Emission Limits under Ecological Uncertainty, in Brennan, M. J. and L. Trigeorgis (eds.), *Project Flexibility, Agency and Competition. New Developments in the Theory and Applications of Real Options,* Oxford University Press, New York, 254-271.

Shah, S. A. (2005), Optimal management of durable pollution, *Journal of Economic Dynamics and Control,* **29**(6), 1121-1164.

Siddiqui, A. and S.-E. Fleten (2010), How to Proceed with Competing Alternative Energy Technologies: A Real Options Analysis, *Energy Economics,* **32**, 817-830.

Stokey, N. L. (2009), *The Economics of Inaction: Stochastic Control models with fixed costs.* Princeton University Press, USA.

Tsujimura, M. (2014), Sequential Investment in Pollution Control Equipment under Uncertainty, *Doshisha Syogaku,* **66**(1), 267-279.

Tsujimura, M. (2015), Pollutant Abatement Investment under Ambiguity in a Two-Period Model, *International Journal of Real Options and Strategy,* **3**, 13-26.

Wirl, F. (2006), Pollution Thresholds under Uncertainty, *Environmental and Development Economics,* **11**, 493-506.

Yang, R-C. and K-H. Liu (2004), Optimal Singular Stochastic Problem on Harvesting System, *Applied Mathematics E-Notes,* **4**, 133-141.

Yong, J. and X. Y. Zhou (1999), *Stochastic Controls: Hamiltonian Systems and HJB Equations,* Springer-Verlag, New York.