Approximating $K(\pi, 3)$ by a Homogeneous Space

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Improving a result of Shastri, we show that if X is a homogeneous space such that $\pi_k(X) = \pi_k(K(\pi, 3))$ for $k \leq 35$, then π is finite.

Key words: abelian group, homogeneous space, Lie group

1. Introduction

As in [3)], W. Browder posed to Shastri a question: given a finitely generated abelian group π and positive integers m, n, is there a homogeneous space X such that

$$\pi_*(X) = \pi_*(K(\pi, n))$$

for $* \leq m$? As is written in [3)], this approximation problem has already been settled for $n > 3$ when Browder posed this question to Shastri, and Shastri studied the remaining case $n = 3$ and proved:

Theorem 1.1 (Shastri [3)]). *If X is a homogeneous space such that*

$$\pi_k(X) = \pi_k(K(\pi, 3))$$

for $k \leq 63$, then π is finite.

In his proof, Shastri only used the assumptions that

$$\pi_k(X) = 0$$

for $0 \leq k \leq 19,\ k \neq 3$ and

$$\pi_k(X) \otimes \mathbb{Z}_{(31)} = 0$$

for $k = 62, 63$. We will considerably reduce the assumptions as:

Theorem 1.2. *If X is a 2-connected homogeneous space such that*

$$\pi_k(X) = 0$$

for $4 \leq k \leq 11$ and

$$\pi_{35}(X) \otimes \mathbb{Q} = 0,$$

then π is finite.

Then we can substantially improve the above result of Shastri.

Corollary 1.3. *If X is a homogeneous space such that*

$$\pi_k(X) = \pi_k(K(\pi, 3))$$

for $k \leq 35$, then $\pi_3(X)$ is finite.

2. Proof of Theorem 1.2

Recall that a simply connected compact simple Lie group is isomorphic with one of

$\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ $(n \geq 2)$, $\mathrm{Spin}(n)$ $(n \geq 7)$,
G_2, F_4, E_6, E_7, E_8

and that a compact simply connected Lie group is a product of finite numbers of compact simply connected simple Lie groups which are unique up to permutation. Then for a compact simply connected Lie group G and a compact simply connected simple Lie group K, we

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can define the integer $m_G(K)$ to be the number of K appearing as the direct product factor of G. Put

$$m_G = \sum_{K\,:\,\text{simple}} m_G(K).$$

Hereafter, let X be a 2-connected homogeneous space. Then there are a compact simply connected Lie group G and its compact simply connected Lie subgroup H such that X is homotopy equivalent to the quotient G/H, so we may assume $X = G/H$. Consider the homotopy exact sequence

$$\cdots \to \pi_{*+1}(X) \to \pi_*(H) \to \pi_*(G) \to \pi_*(X) \to \cdots \tag{2.1.}$$

If K is a compact simply connected simple Lie group, we have $\pi_2(K) = 0$ and $\pi_3(K) \cong \mathbb{Z}$, implying $\pi_3(G) \cong \mathbb{Z}^{m_G}$ and $\pi_3(H) \cong \mathbb{Z}^{m_H}$. Then we get:

Lemma 2.1. *Suppose $\pi_4(X) = 0$. Then the rank of $\pi_3(X)$ is*

$$m_G - m_H.$$

So we aim to prove $m_G = m_H$ under the assumption of Theorem 1.2. By [2)], we have the following tables.

Table 1. list of $\pi_k(G)$.

G	$k=5$	$k=6$	$k=7$	$k=8$
SU(2)	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
SU(3)	0	$\mathbb{Z}/6$	0	$\mathbb{Z}/12$
SU(4)	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/24$
SU(n) ($n \geq 5$)	$\mathbb{Z}$	0	$\mathbb{Z}$	0
Sp(n) ($n \geq 2$)	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0
Spin(n) ($n = 7, 9$)	0	0	$\mathbb{Z}$	$(\mathbb{Z}/2)^2$
Spin(8)	0	0	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^3$
Spin(n) ($n \geq 10$)	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
G_2	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/2$
F_4	0	0	0	$\mathbb{Z}/2$
E_6	0	0	0	$\mathbb{Z}/2$
E_7	0	0	0	0
E_8	0	0	0	0

Table 2. list of $\pi_k(G)$.

G	$k=10$
SU(2)	$\mathbb{Z}/15$
SU(3)	$\mathbb{Z}/30$
SU(4)	$\mathbb{Z}/120 \oplus \mathbb{Z}/2$
SU(5)	$\mathbb{Z}/120$
SU(n) ($n \geq 6$)	0
Sp(2)	$\mathbb{Z}/120$
Sp(n) ($n \geq 3$)	0
Spin(n) ($n = 7, 9$)	$\mathbb{Z}/8$
Spin(8)	$\mathbb{Z}/24 \oplus \mathbb{Z}/8$
Spin(10)	$\mathbb{Z}/4$
Spin(11)	$\mathbb{Z}/2$
Spin(n) ($n \geq 12$)	0
G_2	0
F_4	0
E_6	0
E_7	0
E_8	0

For a compact simply connected simple Lie group K, we put $m(K) = m_G(K) - m_H(K)$.

Lemma 2.2. *If $\pi_k(X) = 0$ for $4 \leq k \leq 11$, then the rank of $\pi_3(X)$ is*

$$m(\mathrm{E}_7) + m(\mathrm{E}_8).$$

Proof. By the homotopy exact sequence (2.1.), we have

$$\pi_k(G) \cong \pi_k(H) \quad (4 \leq k \leq 10).$$

Then by looking at the homotopy groups in the above tables, we can derive equations among $m(K)$. By looking at π_6, we have

$$m(\mathrm{SU}(2)) = m(\mathrm{SU}(3)) = m(\mathrm{G}_2) = 0. \tag{2.2.}$$

By π_5 and (2.2.), we have

$$\sum_{n\geq 2} m(\mathrm{SU}(n)) = 0 \quad \text{and} \quad \sum_{n\geq 2} m(\mathrm{Sp}(n)) = 0. \tag{2.3.}$$

Hence by π_7,

$$m(\mathrm{Spin}(7)) + 2m(\mathrm{Spin}(8)) + \sum_{n\geq 9} m(\mathrm{Spin}(n)) = 0. \tag{2.4.}$$

By π_8, (2.2.), and (2.4.), we have

$$\sum_{n=7}^{9} m(\mathrm{Spin}(n)) + m(\mathrm{F}_4) + m(\mathrm{E}_6) = 0. \tag{2.5.}$$

By π_{10} and (2.2.), we have $m(\mathrm{Spin}(8)) = 0$ and then $m(\mathrm{Spin}(7)) + m(\mathrm{Spin}(9)) = 0$. Combining these equations with (2.4.) and (2.5.), we have

$$\sum_{n\geq 7} m(\mathrm{Spin}(n)) = 0 \tag{2.6.}$$

and

$$m(\mathrm{F}_4) + m(\mathrm{E}_6) = 0, \tag{2.7.}$$

respectively. Thus by (2.2.), (2.3.), (2.6.), and (2.7.), we obtain

$$\begin{aligned} &m_G - m_H \\ =&\sum_{n\geq 2} m(\mathrm{SU}(n)) + \sum_{n\geq 2} m(\mathrm{Sp}(n)) + \sum_{n\geq 7} m(\mathrm{Spin}(n)) \\ &+ m(\mathrm{G}_2) + m(\mathrm{F}_4) + m(\mathrm{E}_6) + m(\mathrm{E}_7) + m(\mathrm{E}_8) \\ =&m(\mathrm{E}_7) + m(\mathrm{E}_8), \end{aligned}$$

completing the proof by Lemma 2.2. □

Proof of Theorem 1.2. Let K be a compact simply connected simple Lie group of rank ≤ 8. If

$$\pi_{35}(K) \otimes \mathbb{Q} \neq 0,$$

then K is either E_7 or E_8. Indeed since

$$\pi_{35}(K) \otimes \mathbb{Q} \cong QH^{35}(K; \mathbb{Q}),$$

by looking at the rational cohomology we see that K is either E_7 or E_8 whenever

$$\pi_{35}(K) \otimes \mathbb{Q} \neq 0,$$

where QA means the module of indecomposables of an algebra A. Then by the homotopy exact sequence (2.1.), if

$$\pi_{35}(X) \otimes \mathbb{Q} = 0,$$

then

$$m(\mathrm{E}_7) + m(\mathrm{E}_8) \leq 0.$$

Thus by Lemma 2.2, the proof is completed. □

References

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