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Eigenvalue Problem in Min-Plus Algebra

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In the present paper, we focus on the eigenvalue problem of matrices with entries in min-plus algebra. Min-plus algebra is one of many idempotent semirings which have been studied in various fields of mathematics. Many of the theorems and techniques that we use in classical linear algebra seems to have analogues in linear algebra over min-plus algebra. However, such kinds of investigation have not yet exploited sufficiently. In the present paper, we try to give a characterization of eigenvalues of matrices with entries in min-plus algebra. We prove that eigenvalues of the weighted adjacency matrix of the network on digraphs with entries in min-plus algebra correspond to the average weight of the directed circuits. However, it is impossible for us to prove that there always exists an eigenvalue of weighted adjacency matrices for an arbitrary circuit of the network. So, the problem to characterize circuits for which there exits eigenvalues that are equal to the average weight of the circuit is left as an open problem.

Key words : Min-Plus algebra, eigenvalue, digraph, circuit

1. Min-Plus Algebra

1.1 Definition and Basic Algebraic Properties

Let $\mathbb{R}_{\min}$ be a set of reals and the element $+\infty$: $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$. The min-plus algebra is an algebra over $\mathbb{R}_{\min}$ with the two binary operations $\oplus$ and $\otimes$:

$$a \oplus b = \min\{a, b\} \ , \ a \otimes b = a + b \ .$$

The operations $\oplus$ and $\otimes$ are associative and commutative for all $a, b, c \in \mathbb{R}_{\min}$:

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c \ , \ a \otimes (b \otimes c) = (a \otimes b) \otimes c$$
$$a \oplus b = b \oplus a \ , \ a \otimes b = b \otimes a \ .$$

In $\mathbb{R}_{\min}$, $\varepsilon = +\infty$ is the identity of $\oplus$:

$$a \oplus \varepsilon = \varepsilon \oplus a = \min\{a, +\infty\} = a \ .$$

The identity of $\otimes$ is $e = 0$:

$$a \otimes e = e \otimes a = a + 0 = a \ .$$

If $x \neq \varepsilon$, there exists an inverse y of x with respect to $\otimes$:

$$x \otimes y = e \ .$$

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The operation $\otimes$ is distributive with respect to $\oplus$:

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z) \ .$$

The identity $\varepsilon = +\infty$ of $\oplus$ is absorbing for $\otimes$:

$$x \otimes \varepsilon = \varepsilon \otimes x = x + \infty = +\infty = \varepsilon \ .$$

The operation $\oplus$ is idenpotent:

$$x \oplus x = \min\{x, x\} = x \ .$$

Definition 1. For $x \in \mathbb{R}_{\min}$ and $k \in \mathbb{N}$, the k^{th} power of x is defined by:

$$x^{\otimes k} = \underbrace{x \otimes x \otimes \ldots \otimes x}_{k \text{ times}} \ .$$

In $\mathbb{R}_{\min}$, the k^{th} power of x reduces to conventional multiplication $x^{\otimes k} = kx$.

It is easy to verify that the min-plus power has the following properties. For $x \in \mathbb{R}_{\min}$, $m, n \in \mathbb{N}$,

1. $x^{\otimes m} \otimes x^{\otimes n} = x^{\otimes (m \otimes n)}$
2. $(x^{\otimes m})^{\otimes n} = x^{\otimes (m^{\otimes n})}$
3. $x^{\otimes 1} = x$
4. $x^{\otimes m} \otimes y^{\otimes m} = (x \otimes y)^{\otimes m}$.

1.2 Matrices in Min-Plus Algebra

Here we will discuss on matrices with entries in $\mathbb{R}_{\min}$. For $m, n \in \mathbb{N}$, let $\mathbb{R}_{\min}^{m\times n}$ be a set of all $m \times n$ matrices in $\mathbb{R}_{\min}$. We will define the several operations in $\mathbb{R}_{\min}^{m\times n}$ as follows.

Definition 2.

1. For $A, B \in \mathbb{R}_{\min}^{m\times n}$, their sum $A \oplus B \in \mathbb{R}_{\min}^{m\times n}$ is defined by:
$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \min\{a_{ij}, b_{ij}\} .$$
2. For $A \in \mathbb{R}_{\min}^{m\times k}$ and $B \in \mathbb{R}_{\min}^{k\times n}$, their product $A \otimes B \in \mathbb{R}_{\min}^{m\times n}$ is defined by:
$$(A \otimes B)_{ij} = \bigoplus_{\ell=1}^{k} (a_{i\ell} \otimes b_{\ell j}) = \min_{\ell=1,2,\ldots,k} \{a_{i\ell} + b_{\ell j}\}$$
3. For $A \in \mathbb{R}_{\min}^{m\times n}$, the transpose matrix ${}^tA$ of A is defined as in conventional algebra $({}^tA)_{ij} = (A)_{ji}$.
4. The $n \times n$ identity matrix I_n is defined by
$$(I_n)_{ij} = \begin{cases} e \text{ if } i = j \\ \varepsilon \text{ if } i \neq j \end{cases}$$
Then we can see that $A \otimes I_n = I_n \otimes A = A$ for $A \in \mathbb{R}_{\min}^{n\times n}$. We will abbreviate $I_n = I$ when the order of a matrix is clear from the context.
5. For $A \in \mathbb{R}_{\min}^{n\times n}$ and $k \in \mathbb{N}$, the k^{th} power of A is defined by:
$$A^{\otimes k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}} .$$
If $k = 0$, we set $A^{\otimes 0} = I$.
6. For $A \in \mathbb{R}_{\min}^{m\times n}$ and $\alpha \in \mathbb{R}_{\min}$, their product $\alpha \otimes A \in \mathbb{R}_{\min}^{m\times n}$ is defined by:
$$(\alpha \otimes A)_{ij} = \alpha \otimes (A)_{ij} .$$

The operation $\oplus$ is commutative in $\mathbb{R}_{\min}^{m\times n}$, but $\otimes$ is not. The operation $\otimes$ is distributive with respect to $\oplus$ in matrices. Also $\oplus$ is idempotent in $\mathbb{R}_{\min}^{m\times n}$, that is, we have $A \oplus A = A$.

2. Graph Theory and Min-Plus Algebra

2.1 Digraph

A directed graph or, for short, a digraph G is a finite nonempty set V of elements called vertices together with a set E of ordered pair of V called edges. To indicate that a digraph G consists of the vertex set V and the edge set E, we write $G = (V, E)$. Each elements $e \in E$ can be expressed as (u, v) with $u, v \in V$. In the present paper, we consider the digraph $G = (V, E)$ with set of vertices $V = \{v_1, \ldots, v_n\}$ and the set of edges $E = \{e_1, \ldots, e_m\}$. We introduce the maps $\partial^- : E \to V$ and $\partial^+ : E \to V$ by $\partial^-(e) = u$, $\partial^+(e) = v$ for $e = (u, v)$, respectively. In this case, the vertices u and v are called the tail and the head of e, respectively, and they are simply called the end vertices of e. If distinct edges e and e' have two end vertices in common, then (i)$\partial^-(e) = \partial^-(e')$, $\partial^+(e) = \partial^+(e')$ or (ii)$\partial^-(e) = \partial^+(e')$, $\partial^+(e) = \partial^-(e')$ occurs. In the first case e and e' are called parallel edges and the second case, they are called antiparallel edges. An edge with just one end vertex ($\partial^-(e) = \partial^+(e)$) is called a loop. A graph without loops, parallel edges and antiparallel edges is called simple. A path P in G is an alternating sequence of vertices and edges $P = (v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_s}, v_{i_s})$ such that each e_{i_j} has the tail $v_{i_{j-1}}$ and the head v_{i_j}, and the vertices and the edges are pairwise distinct except the vertices v_{i_0}, v_{i_s}. If $P = (v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_s}, v_{i_s})$ is a path, then v_{i_0} and v_{i_k} are called the initial and the terminal vertex, respectively, and a path P from v_{i_0} to v_{i_s} is expressed as a v_{i_0}-v_{i_s} path. The path P is called closed if its initial and terminal vertex coincide; the closed path is called a circuit. We consider a pair of parallel edges and a pair of antiparallel edges as circuits.

Definition 3. Let G be a digraph with n vertices and m edges. We define the adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ of G by:
$$a_{ij} = \begin{cases} 1 & \text{if} \quad (v_i, v_j) \in E \\ 0 & \text{if} \quad (v_i, v_j) \notin E \end{cases}$$

2.2 Network

Let $G = (V, E)$ be a digraph with n vertices and m edges. We assign the integer $w(e)$ to each edge $e \in E$; $w(e)$ is called the weight of the edge e. The pair $\mathcal{N} = (G, w)$ is called a network on the digraph G endowed with the weight w.

Definition 4. Let $\mathcal{N}$ be a network on the digraph G. We define the weighted adjacency matrix $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ of $\mathcal{N}$ by:

$$b_{ij} = \begin{cases} w(e) & \text{if} \quad e = (v_i, v_j) \in E \\ 0 & \text{if} \quad e = (v_i, v_j) \notin E \end{cases}$$

Let $P = (v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_s}, v_{i_s})$ be a path in $\mathcal{N}$. The length $\ell(P) = s$ of the path P is the number of edges in P; the weight $\omega(P)$ of the path P is defined by:

$$\omega(P) = \sum_{k=0}^{s-1} b_{i_k i_{k+1}} .$$

We define the length $\ell(C)$ and the weight $\omega(C)$ of a circuit C in the same way as the difinition for paths.

Definition 5. The average weight of a circuit C is defined by $\frac{\omega(C)}{\ell(C)}$.

2.3 Network in Min-Plus Algebra

Let $G = (V, E)$ be a digraph with n vertices and m edges and $\mathcal{N} = (G, w)$ be a network on G. First we redefine the weight $w' : E \to \mathbb{R}_{\min}$ with values in $\mathbb{R}_{\min}$ in terms of the weight w as follows: $w'(e) = w(e)$ for $w(e) \neq 0$ and $w'(e) = \varepsilon$ for $w(e) = 0$. Then we define the network $\mathcal{N}'$ with values in $\mathbb{R}_{\min}$ on the digraph G by the pair $\mathcal{N}' = (G, w')$ of G and w'. Next we define the weighted adjacency matrix of the network $\mathcal{N}'$ with values in $\mathbb{R}_{\min}$ as follows.

Definition 6. Let $\mathcal{N}'$ be a network on the digraph G with values in $\mathbb{R}_{\min}$. We define the weighted adjacency matrix $B' = (b'_{ij}) \in \mathbb{R}_{\min}^{n\times n}$ of $\mathcal{N}'$ by:

$$b'_{ij} = \begin{cases} w(e) & \text{if} \quad e = (v_i, v_j) \in E \\ +\infty & \text{if} \quad e = (v_i, v_j) \notin E \end{cases}$$

Since the conventional addition + becomes the operatin $\otimes$ in $\mathbb{R}_{\min}$, we compute the weight $\omega(P)$ of the path $P = (v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_s}, v_{i_s})$ in the digraph G as follows:

$$\omega(P) = \bigotimes_{k=0}^{s-1} b'_{i_k i_{k+1}} .$$

3. Eigenvalue in Min-Plus Algebra

Here we will discuss on the eigenvalues of matrices with entries in min-plus algebra. We show that the min-plus eigenvalues have a graph theoretical interpretation.

Definition 7. Given a matrix $A \in \mathbb{R}_{\min}^{n\times n}$, we say that $\lambda \in \mathbb{R}_{\min}$ is an eigenvalue of A if there exists $\boldsymbol{x} \in \mathbb{R}_{\min}^{n}$ such that $\boldsymbol{x} \neq {}^t(\varepsilon, \varepsilon, \ldots, \varepsilon)$ and:

$$A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x} .$$

The vector $\boldsymbol{x}$ is referred to as the eigenvector of A for the eigenvalue λ.

The definition above also allows the eigenvalue to be equal to ε and the eigenvector to have entries equal to ε. We have the following proposition.

Proposition 8. The identity ε of $\oplus$ is an eigenvalue of A if and only if A has a column whose all entries are ε.

proof. Let $\boldsymbol{x}$ be an eigenvector of A for the eigenvalue $\lambda = \varepsilon$. By definition, the eigenvector $\boldsymbol{x}$ has at least one element which is not equal to ε such that $x_j \neq \varepsilon$. Then we will prove that all entries of the j^{th} column of A is equal to ε. If one entry of the j^{th} column of A has a value a_{ij} in i^{th} row, then we see $a_{ij} \otimes x_j \neq \varepsilon$. On the other hand, $\lambda \otimes x_i = \varepsilon$ since $\lambda = \varepsilon$. Therefore A has a column whose all entries are ε. □

For a matrix $A \in \mathbb{R}_{\min}^{n\times n}$, we can construct the network $\mathcal{N}' = (G, w')$ whose weighted adjacency matrix with entries in $\mathbb{R}_{\min}$ is identical with A. To indicate that $\mathcal{N}'$ is determined by the matrix A, we write $\mathcal{N}' = \mathcal{N}(A) = (G(A), w')$. The network $\mathcal{N}(A)$ with values in $\mathbb{R}_{\min}$ is called the network associated with the matrix $A \in \mathbb{R}_{\min}^{n\times n}$. We give one characterization of eigenvalues of matrices $A \in \mathbb{R}_{\min}^{n\times n}$ which are not equal to ε in the followg theorem.

Theorem 9. Any eigenvalue $\lambda(\neq \varepsilon)$ of a squre matrix A become the average weight of some circuit in $\mathcal{N}(A)$.

proof. For convenience, we assume the set of vertices of the graph associated with the matrix A as $V = \{1, 2, \ldots, n\}$. Then set of edges E consists of pairs of vertices as (i, j) with suitable $i, j \in V$. Further in the proof, we express paths and circuits as the sequece of edges for abbreviation. By definition, an eigenvector $\boldsymbol{x}$ of λ has at least one element which is not equal to ε. This means that there exists μ_1 such that $x_{\mu_1} \neq \varepsilon$. Therefore h $[A \otimes \boldsymbol{x}]_{\mu_1} = \lambda \otimes x_{\mu_1} \neq \varepsilon$. Hence we can find a vertex μ_2 with $a_{\mu_1\mu_2} \otimes x_{\mu_2} = \lambda \otimes x_{\mu_1}$, implying that $a_{\mu_1\mu_2} \neq \varepsilon$, $x_{\mu_2} \neq \varepsilon$ and $(\mu_1, \mu_2) \in E$. Similarly there exists μ_3 such that $a_{\mu_2\mu_3} \otimes x_{\mu_3} = \lambda \otimes x_{\mu_2}$ with $(\mu_2, \mu_3) \in E$. Applying the same procedure as above, we find the vertex μ_h that we encounter twice for the first time, then we see $\mu_h = \mu_{h+k}$, since the number of vertices is finite. So we have found a circuit C:

$$C = ((\mu_h, \mu_{h+1}), (\mu_{h+1}, \mu_{h+2}), \ldots (\mu_{h+k-1}, \mu_h))$$

This has the length $\ell(C) = k$ and the weight $\omega(C) = \bigotimes_{j=0}^{k-1} a_{\mu_{h+j}\mu_{h+j+1}}$, where $\mu_h = \mu_{h+k}$. By construction, we have

$$\bigotimes_{j=0}^{k-1}(a_{\mu_{h+j}\mu_{h+j+1}} \otimes x_{\mu_{h+j+1}}) = \lambda^{\otimes k} \otimes \bigotimes_{j=0}^{k-1} x_{\mu_{h+j}}$$

Since $\otimes$ converts to $+$ in conventional algebra the equation above can be written as

$$\sum_{j=0}^{k-1}(a_{\mu_{h+j}\mu_{h+j+1}} + x_{\mu_{h+j+1}}) = k\lambda + \sum_{j=0}^{k-1} x_{\mu_{h+j}}$$

We also have that

$$\sum_{j=0}^{k-1} x_{\mu_{h+j+1}} = \sum_{j=0}^{k-1} x_{\mu_{h+j}}$$

since $\mu_h = \mu_{h+k}$. Using this fact, we can substruct $\sum_{j=0}^{k-1} x_{\mu_{h+j}}$ from both sides giving us,

$$\bigotimes_{j=0}^{k-1} a_{\mu_{h+j}\mu_{h+j+1}} = k\lambda$$

This means that $w(C) = k\lambda$. Then the average weight of the circuit C is

$$\frac{\omega(C)}{\ell(C)} = \frac{k\lambda}{k} = \lambda$$

$\square$

Reference

1) D. Jungnickel: *Graphs, Networks and Algorithm*, Second Edition, Springer Verlag, Berlin Heidelberg, 2005.

2) K. Ahuja, L. Magnanti and B. Orlin: *NETWORK FLOWS*, Prentice-Hall, New Jersey, 1993.

3) U. Zimmermann: *Linear and Combinatorial Optimization in Ordered Algebraic Structures*, North-Holland Publishing Company, 1947.

4) M. Gondran and M. Minoux: *Graph, Dioids and Semiring*, Springer Verlag, Berlin Heidelberg, 2008.