Two-step Family of "Look-ahead" Linear Multistep Method for ODEs

Taketomo MITSUI* and Dauda Gulibur YAKUBU**

(Received July 4, 2011)

A family of two-step "look-ahead" linear multistep methods (LALMM) is formulated and analysed. LALMM is a new class of discrete variable methods for numerical solution of initial-value problem of ordinary differential equations (ODEs). Two fourth-order pairs of predictor and corrector are derived and showed to be $A(\theta)$-stable. The obtained schemes provide the basis for futher developement of LALMM.

Key words : numerical solution, discrete variable method, predictor-corrector iteration, convergency, stability

1. Introduction and motivation

We are concerned with numerical solutions of initial-value problem of ordinary differential equations (ODEs) given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \quad (a \le x \le b), \quad y(a) = y_I. \tag{1}$$

Here the unknown function y is a mapping $[a, b] \to \mathbb{R}^d$, the right-hand side function f is $[a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ and the initial vector y_I is given in $\mathbb{R}^d$. Among many numerical solutions, we are interested in the discrete variable methods (DVMs) with a constant step-size h to generate the approximate solution y_n of (1) on the step-point $x_n = a+nh$ (the book[1] is an excellent reference of the topic). In the previous work[2], a new class of DVMs was proposed as "look-ahead" linear multistep methods (LALMM). The general framework of LALMM is as follows.

Assume that we look for the numerical solution of the $(n + k)$-th step-point when the back-values $y_n, y_{n+1}, \ldots, y_{n+k-1}$ and a pre-assigned initial guess $y^{[0]}_{n+k}$ are available. First, we look ahead for the $(n + k + 1)$-st step-point by

$$y^{[0]}_{n+k+1} + \alpha_k y^{[0]}_{n+k} + \sum_{i=0}^{k-1} \alpha_i y_{n+i} = h\left(\beta_k f(x_{n+k}, y^{[0]}_{n+k}) + \sum_{i=0}^{k-1} \beta_i f(x_{n+i}, y_{n+i})\right), \tag{2}$$

which can be regarded as a predictor. Then, correct the look-for value by

$$y^{[1]}_{n+k} + \sum_{i=0}^{k-1} \alpha^*_i y_{n+i} = h\left(\beta^*_{k+1} f(x_{n+k+1}, y^{[0]}_{n+k+1}) + \beta^*_k f(x_{n+k}, y^{[0]}_{n+k}) + \sum_{i=0}^{k-1} \beta^*_i f(x_{n+i}, y_{n+i})\right). \tag{3}$$

When a (local) convergence attains, *i.e.,* the estimation

$$\|y^{[1]}_{n+k} - y^{[0]}_{n+k}\| \le \delta_{TOL}$$

holds for a pre-assigned error tolerance δ_{TOL}, we complete the current step and advance to the next step. Otherwise, we replace $y^{[0]}_{n+k}$ by $y^{[1]}_{n+k}$ and iterate (2) and (3). Note that generally we assume $\alpha_j \ne \alpha^*_j, \beta_j \ne \beta^*_j$.

* Department of Mathematical Sciences, Faculty of Science & Engineering, Doshisha University, Kyoto 610-0394
Phone: +81-774-65-6689, E-mail: tamitsui@mail.doshisha.ac.jp
** Mathematical Sciences Programme, Abubakar Tafawa Balewa University, Bauchi, Nigeria

In the class of LALMM, the already known pairs are the cases with $k = 1$[3, 4], which is essentially single-step, or with k more than four[5]. As a test-bed of LALMM, a two-step ($k = 2$) family should be developed and practiced. The present note aims the purpose. By introducing a convenient and powerful technique employing the the power series expansion of $\log(1+\xi)$, we derive several possible schemes of two-step LALMM with the aimed order of accuracy and analyse their stability.

2. Corrector equation

According to Theorem 1 of the previous paper[2], the order of accuracy of the predictor of an LALMM pair can be one less than that of the corrector for the global convergence of the pair. Thus, first we will analyse the corrector equation.

The Theorem also states that for the convergence the first characteristic polynomial of corrector $\rho^*(\zeta) = \zeta^k + \sum_{i=0}^{k-1} \alpha_i^* \zeta^i$ should be restricted to be $\alpha_\ell^* = -1$ for a certain ℓ and other α^*'s zero. Thereofore in the two-step case we assume the corrector in the form:

$$y_{n+2} + \alpha_1^* y_{n+1} + \alpha_0^* y_n = h\left(\beta_3^* f_{n+3} + \beta_2^* f_{n+2} + \beta_1^* f_{n+1} + \beta_0^* f_n\right) \quad (4)$$

We will consider two types of corrector.

Adams type $\alpha_1^* = -1, \alpha_0^* = 0$

Milne type $\alpha_1^* = 0, \alpha_0^* = -1$

Note that either case satisfies $1 + \alpha_1^* + \alpha_0^* = 0$, one of the consistency conditions for the corrector ((15) of the previous work[2]).

2.1 Adams-type corrector

The first and second characteristic polynomials are

$$\rho^*(\zeta) = \zeta^2 - \zeta, \qquad \sigma^*(\zeta) = \beta_2^*\zeta^2 + \beta_1^*\zeta + \beta_0^* \quad (5)$$

We introduce the shift operator S with the reference step-size h and express the corrector equation as:

$$\rho^*(S) y_n = h(\beta_3^* S^3 + \sigma^*(S)) f_n.$$

Furthermore, an introduction of the the differentiation operator D yields

$$\begin{aligned} &\rho^*(S)y(x) - (\beta_3^* S^3 + \sigma^*(S))hDy(x) \\ &\quad = \widetilde{K}_q (hD)^{q+1} y(x) + O(h^{q+2}) \end{aligned} \quad (6)$$

for an infinitely differentiable function $y(x)$. Here, the index q, the term $\widetilde{K}_q (hD)^{q+1} y(x)$ and the constant $\widetilde{K}_q$ mean the order of accuracy, the local truncation error and the error constant, respectively, of the corrector. Since the equation

$$Sy(x) = \exp(hD)y(x)$$

holds, Eq. (6) can be formally expressed as:

$$\rho^*(S) - (\beta_3^* S^3 + \sigma^*(S)) \log S = \widetilde{K}_q \log^{q+1} S + O(h^{q+2}) \quad (7)$$

This is the key equation which derives q and $\widetilde{K}_q$.

To analyse the order, we note that the formal expression $S = 1+\xi$ ($\xi \in \mathbb{R}, |\xi| < 1$) is useful. This means from (7) we have

$$\begin{aligned} &\rho^*(1+\xi) - (\beta_3^*(1+\xi)^3 + \sigma^*(1+\xi)) \log(1+\xi) \\ &\quad = \widetilde{K}_q \log^{q+1}(1+\xi) + O(\xi^{q+2}). \end{aligned} \quad (8)$$

Note that the identities $\rho^*(1+\xi) = \xi(1+\xi)$ and

$$\begin{aligned} \log(1+\xi) &= \xi\left(1 - \tfrac{1}{2}\xi + \tfrac{1}{3}\xi^2 - \tfrac{1}{4}\xi^3 + \cdots\right) \\ &= \xi \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1} \xi^r \end{aligned} \quad (9)$$

hold (As for this way of analysis, one may refer to Urabe's work[6]). We introduce new coefficients μ_i^* by

$$\beta_3^*(1+\xi)^3 + \sigma^*(1+\xi) = \mu_3^*\xi^3 + \mu_2^*\xi^2 + \mu_1^*\xi + \mu_0^*,$$

which yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_3^* \\ \beta_2^* \\ \beta_1^* \\ \beta_0^* \end{bmatrix} = \begin{bmatrix} \mu_3^* \\ \mu_2^* \\ \mu_1^* \\ \mu_0^* \end{bmatrix}.$$

Therefore, (8) implies

$$1+\xi-(\mu_0^*+\mu_1^*\xi+\mu_2^*\xi^2+\mu_3^*\xi^3)\times\left(1-\tfrac{1}{2}\xi+\tfrac{1}{3}\xi^2-\tfrac{1}{4}\xi^3+\cdots\right)=\widetilde{K}_q\log^q(1+\xi)+O(\xi^{q+1}), \tag{10}$$

which gives q and $\widetilde{K}_q$ by comparing the same power of ξ in both sides. In fact, we have the conditions

$$1-\mu_0^*=0, \tag{11}$$

$$1-\left(\mu_1^*-\frac{1}{2}\mu_0^*\right)=0, \tag{12}$$

$$\mu_2^*-\frac{1}{2}\mu_1^*+\frac{1}{3}\mu_0^*=0, \tag{13}$$

$$\mu_3^*-\frac{1}{2}\mu_2^*+\frac{1}{3}\mu_1^*-\frac{1}{4}\mu_0^*=0, \tag{14}$$

which give $q=4$ and

$$\widetilde{K}_4=\frac{1}{2}\mu_3^*-\frac{1}{3}\mu_2^*+\frac{1}{4}\mu_1^*-\frac{1}{5}\mu_0^*.$$

Note that (11) is equivalent to the second condition of consistency of the corrector. Eqs. (11) to (14) have the unique solution

$$\mu_0^*=1,\quad \mu_1^*=\frac{3}{2},\quad \mu_2^*=\frac{5}{12},\quad \mu_3^*=-\frac{1}{24},$$

which implies $\widetilde{K}_4=11/720$. Converting into β_i^*, we have

$$\beta_0^*=-\frac{1}{24},\quad \beta_1^*=\frac{13}{24},\quad \beta_2^*=\frac{13}{24},\quad \beta_3^*=-\frac{1}{24}$$

Consequently, we derive the Adams-type corrector given by

$$y_{n+2}-y_{n+1}=\frac{h}{24}\left(-f_{n+3}+13f_{n+2}+13f_{n+1}-f_n\right), \tag{15}$$

which is of fourth order and the error constant $\widetilde{K}_4=11/720$.

Milne type corrector will be reported later.

3. Predictor equation

Since the corrector has been obtained as of fourth order, predictor is sufficient to be of third order. Assume the predictor equation

$$y_{n+3}+\alpha_2y_{n+2}+\alpha_1y_{n+1}+\alpha_0y_n=h\left(\beta_2f_{n+2}+\beta_1f_{n+1}+\beta_0f_n\right), \tag{16}$$

whose first and second characteristic polynomials are given by

$$\begin{cases}\rho(\zeta)=\alpha_2\zeta^2+\alpha_1\zeta+\alpha_0,\\ \sigma(\zeta)=\beta_2\zeta^2+\beta_1\zeta+\beta_0.\end{cases} \tag{17}$$

A formal expression of (16) with the shift operator is

$$\left(S^3+\rho(S)\right)y_n=h\sigma(S)f_n. \tag{18}$$

Similar to the case of the corrector, the order p and the error constant K_p are defined through

$$\left\{S^3+\rho(S)\right\}-\sigma(S)\log S=K_p\log^{p+1}S+O(h^{p+2}),$$

which is equivalent to

$$(1+\xi)^3+\rho(1+\xi)-\sigma(1+\xi)\log(1+\xi)=K_p\log^{p+1}(1+\xi)+O(\xi^{p+2}). \tag{19}$$

Since the condition $1+\rho(1)=0$ should be satisfied for consistency, we assume

$$(1+\xi)^3+\rho(1+\xi)=\xi(\xi^2+\lambda_1\xi+\lambda_0)$$

where $\lambda_1=\alpha_2+3, \lambda_0=2\alpha_2+\alpha_1+3$. Note that α_0 will be determined by the condition $1+\rho(1)=1+\alpha_2+\alpha_1+\alpha_0=0$. Again we introduce new coefficients μ_i by

$$\sigma(1+\xi)=\mu_2\xi^2+\mu_1\xi+\mu_0,$$

which derives

$$\mu_2=\beta_2,\quad \mu_1=2\beta_2+\beta_1,\quad \mu_0=\beta_2+\beta_1+\beta_0.$$

Consequently we have the equation for order determination as

$$\xi^2+\lambda_1\xi+\lambda_0-\left(\mu_2\xi^2+\mu_1\xi+\mu_0\right)\times\left(1-\frac{1}{2}\xi+\frac{1}{3}\xi^2-\frac{1}{4}\xi^3+\cdots\right)=K_p\xi^p(1+O(\xi)). \tag{20}$$

3.1 Fifth-order predictor

Full balancing of the powers of ξ up to fifth order in each side yields the equations:

$$\lambda_0 - \mu_0 = 0, \tag{21}$$

$$\lambda_1 - \mu_1 + \frac{1}{2}\mu_0 = 0, \tag{22}$$

$$1 - \left(\mu_2 - \frac{1}{2}\mu_1 + \frac{1}{3}\mu_0\right) = 0, \tag{23}$$

$$\frac{1}{2}\mu_2 - \frac{1}{3}\mu_1 + \frac{1}{4}\mu_0 = 0, \tag{24}$$

$$\frac{1}{3}\mu_2 - \frac{1}{4}\mu_1 + \frac{1}{5}\mu_0 = 0. \tag{25}$$

Eqs. (21) to (25) imply the unique solution

$$\lambda_0 = 30, \quad \lambda_1 = 21, \quad \mu_0 = 30, \quad \mu_1 = 36, \quad \mu_2 = 9,$$

which yields

$$\alpha_0 = -10, \quad \alpha_1 = -9, \quad \alpha_2 = 18,$$
$$\beta_0 = 3, \quad \beta_1 = 18, \quad \beta_2 = 9, \tag{26}$$

and the error constant $K_5 = \mu_2/4 - \mu_1/5 + \mu_0/6 = 1/20$. The derived predictor equation is

$$y_{n+3} + 18y_{n+2} - 9y_{n+1} - 10y_n = h\left(9f_{n+2} + 18f_{n+1} + 3f_n\right). \tag{27}$$

3.2 Third-order predictor

In order to make the predictor of third-order accuracy, merely Eqs. (21) to (23) should be satisfied. Then, by taking μ_1 and μ_2 as free parameters, other coefficients are determined by

$$\lambda_0 = \tfrac{3}{2}\mu_1 - 3\mu_2 + 3,$$
$$\lambda_1 = \tfrac{1}{4}\mu_1 + \tfrac{3}{2}\mu_2 - \tfrac{3}{2},$$
$$\mu_0 = \tfrac{3}{2}\mu_1 - 3\mu_2 + 3.$$

The coefficients of the characteristic polynomials are given with two parameters μ_1 and μ_2 by

$$\begin{cases} \alpha_0 = \dfrac{1}{4}(-7\mu_1 + 6\mu_2 + 14), \quad \alpha_1 = 2\mu_1 - 9, \\ \alpha_2 = -\dfrac{1}{4}(\mu_1 + 6\mu_2 - 18), \\ \beta_0 = \dfrac{1}{2}(\mu_1 - 4\mu_2 + 6), \\ \beta_1 = \mu_1 - 2\mu_2, \quad \beta_2 = \mu_2. \end{cases} \tag{28}$$

One of the options to fix the coefficients is to make $\alpha_1 = \beta_1 = 0$. This means $\mu_1 = 9/2$ and $\mu_2 = 9/4$, and leads to the predictor equation in a rather simple form

$$y_{n+3} - y_n = \frac{h}{4}\left(9f_{n+2} + 3f_n\right) \tag{29}$$

with the error constant $K_3 = 3/8$.

Another option is $\alpha_0 = \beta_0 = 0$. This implies $\mu_1 = 46/11$ and $\mu_2 = 28/11$, and leads to another predictor equation

$$y_{n+3} - \frac{4}{11}y_{n+2} - \frac{7}{11}y_{n+1} = \frac{h}{11}(28f_{n+2} - 10f_{n+1}) \tag{30}$$

with the error constant $K_3 = 19/66$. Note that this is essentially an explicit two-step LMM.

4. Derived schemes

LALMM of two-step is carried out in the following procedure.

Initialisation Given the the equation (1) and the step-points $\{x_n\}$ $(n = 0, 1, \ldots;\ x_0 = a, x_N = b)$ with the step-size h. By utilizing the Heun method *twice* from x_0, we compute the starting value y_1 and the predicted value $y_2^{[0]}$. That is, for $n = 0$ and 1 we apply

$$k_1 = f(x_n, y_n),$$
$$k_2 = f(x_n + h/3, y_n + hk_1/3),$$
$$k_3 = f(x_n + 2h/3, y_n + 2hk_2/3),$$
$$y_{n+1} = y_n + (h/4)(k_1 + 3k_3)$$

to obtain y_1 and $y_2^{[0]}$.

Stepping forward Starting from $n = 0$ till $n = N - 2$ we repeat the following steps.

1. Utilizing the back-values $y_n, y_{n+1}, f_n, f_{n+1}$ and the initial guess $y_{n+2}^{[0]}$, we compute the look-ahead value $y_{n+3}^{[0]}$ by (29), *i.e.*,

$$y_{n+3}^{[0]} = y_n + \frac{h}{4}\left(9f(x_{n+2}, y_{n+2}^{[0]}) + 3f_n\right). \tag{31}$$

2. We correct the look-for value y_{n+2} by (15), *i.e.,*

$$y^{[1]}_{n+2} = y_{n+1} + \tfrac{h}{24}\Big(-f(x_{n+3}, y^{[0]}_{n+3}) + 13f(x_{n+2}, y^{[0]}_{n+2}) + 13f_{n+1} - f_n\Big). \quad (32)$$

3. For a pre-assigned error tolerance δ_{TOL}, we test the local convergence by

$$\|y^{[1]}_{n+2} - y^{[0]}_{n+2}\| \le \delta_{TOL}.$$

   When it holds, we terminate the present step and proceed to the next step by increasing n by 1. When the local convergence does not attain, replacing $y^{[0]}_{n+2}$ by the present $y^{[1]}_{n+2}$, we get back to Step 1.

We will call the above procedure as scheme of Adams type-A. When (31) is replaced with that by (30), *i.e.,*

$$y_{n+3} = \frac{4}{11}y(x_{n+2}, y^{[0]}_{n+2}) + \frac{7}{11}y_{n+1} + \frac{h}{11}\Big(28f(x_{n+2}, y^{[0]}_{n+2}) - 10f_{n+1}\Big), \quad (33)$$

it will be called Adams type-B.

5. Stability

The previous work[2)] (Eq. (26) there) shows that the stability polynomial of LALMM is given by

$$\pi(\zeta; z) = \rho^*(\zeta) - z\sigma^*(\zeta) + \beta^*_{k+1}z\rho(\zeta) - \beta^*_{k+1}z^2\sigma(\zeta). \quad (34)$$

The scheme of Adams type-A has the characteristic polynomials as

$$\rho(\zeta) = -1, \quad \sigma(\zeta) = \frac{1}{4}(9\zeta^2 + 3),$$
$$\rho^*(\zeta) = \zeta^2 - \zeta, \quad \sigma^*(\zeta) = \frac{1}{24}(13\zeta^2 + 13\zeta - 1)$$

and $\beta^*_3 = -1/24$. Substitution of these results into (34) derives

$$\pi(\zeta; z) = p_2(z)\zeta^2 + p_1(z)\zeta + p_0(z),$$

where

$$\begin{cases} p_2(z) = 1 - \dfrac{13}{24}z + \dfrac{3}{32}z^2, \\ p_1(z) = -\left(1 + \dfrac{13}{24}z\right), \\ p_0(z) = z\left(\dfrac{1}{12} + \dfrac{1}{32}z\right). \end{cases} \quad (35)$$

Since $\pi(\zeta; z)$ is of second degree with respect to ζ, Schur criterion tells that it is a Schur polynomial, *i.e.,* a polynomial whose roots are all less than unity in magnitude, if and only if

$$|\overline{p_2(z)}| > |p_0(z)| \quad (36)$$

and $\pi_1(\zeta; z)$ is Schur, where $\pi_1(\zeta; z)$ is the reduced polynomial from $\pi(\zeta; z)$ given by

$$\begin{aligned} \widehat{\pi}(\zeta; z) &= \overline{p_0(z)}\zeta^2 + \overline{p_1(z)}\zeta + \overline{p_0(z)}, \\ \pi_1(\zeta; z) &= \frac{1}{\zeta}\left(\widehat{\pi}(0; z)\pi(\zeta; z) - \pi(0; z)\widehat{\pi}(\zeta; z)\right). \end{aligned}$$

Since $\pi_1(\zeta; z)$ is linear in ζ, a simple calculation gives the latter condition is equivalent to

$$|p_0(z)\overline{p_1(z)} - \overline{p_2(z)}p_1(z)| < |p_2(z)|^2 - |p_0(z)|^2. \quad (37)$$

When we express $z = x + \mathrm{i}\,y$ $(x, y \in \mathbb{R})$, the two conditions imply that for (x, y) the inequalities

$$\begin{aligned} P(x, y) &\equiv |\overline{p_2(z)}|^2 - |p_0(z)|^2 \\ &= 1 - \frac{13}{12}x + \frac{1477}{3184}x^2 + \frac{283}{3184}y^2 - \frac{41}{384}x^3 \\ &\quad - \frac{41}{384}xy^2 + \frac{1}{128}x^4 + \frac{1}{64}x^2y^2 + \frac{1}{128}y^4 > 0 \end{aligned} \quad (38)$$

and

$$\begin{aligned} &Q(x, y) \\ &\equiv (|p_2(z)|^2 - |p_0(z)|^2)^2 - |p_0(z)\overline{p_1(z)} - \overline{p_2(z)}p_1(z)|^2 \\ &= -2\,x - \frac{219017}{3667968}xy^2 + \frac{147590111}{729925632}x^2y^2 \\ &- \frac{466591}{3667968}x^3y^2 - \frac{172867}{7335936}xy^4 + \frac{233345}{7335936}x^4y^2 \\ &+ \frac{20377}{1833984}x^2y^4 - \frac{41}{8192}x^5y^2 - \frac{41}{8192}x^3y^4 \\ &- \frac{41}{24576}xy^6 + \frac{1}{4096}x^6y^2 + \frac{3}{8192}x^4y^4 \\ &+ \frac{1}{4096}x^2y^6 + \frac{16853}{6368}x^2 - \frac{385}{19104}y^2 - \frac{4863677}{3667968}x^3 \\ &+ \frac{571665479}{1459851264}x^4 - \frac{2763145}{1459851264}y^4 - \frac{760315}{7335936}x^5 \\ &+ \frac{7133}{407552}x^6 - \frac{23443}{7335936}y^6 - \frac{41}{24576}x^7 \\ &+ \frac{1}{16384}x^8 + \frac{1}{16384}y^8 > 0 \end{aligned} \quad (39)$$

should be held. It is easily seen that $P(x,y) > 1$ holds for all (x,y) $(x<0)$, *i.e.,* z for $\Re z < 0$, because of the identity

$$P(-x,y) = 1 + \frac{13}{12}x + \frac{1477}{3184}x^2 + \frac{283}{3184}y^2 + \frac{41}{384}x^3 + \frac{41}{384}xy^2 + \frac{1}{128}x^4 + \frac{1}{64}x^2y^2 + \frac{1}{128}y^4.$$

On the other hand, the condition $Q(x,0) > 0$ is not necessarily apparent. On the negative real line $Q(x,y) > 0$ holds, for

$$Q(-x,0) = 2x + \frac{16853}{6368}x^2 + \frac{4863677}{3667968}x^3 + \frac{571665479}{1459851264}x^4 + \frac{760315}{7335936}x^5 + \frac{7133}{407552}x^6 + \frac{41}{24576}x^7 + \frac{1}{16384}x^8.$$

However, around the origin of the imaginary line $Q(0,y) \le 0$ holds because of the identity

$$Q(0,y) = -\frac{385}{19104}y^2 - \frac{2763145}{1459851264}y^4 - \frac{23443}{7335936}y^6 + \frac{1}{16384}y^8,$$

which also shows that $Q(0,y)$ becomes positive when y^2 is getting considerably large. Therefore, by a help of the mathematical software, *Maple*, we draw the contour line $Q(x,y) = 0$ on the complex plane and obtain Fig. 1. By these observations we can conclude that the scheme of Adams type-A is $A(\theta)$-stable with a certain positive angle θ.

The scheme of Adams type-B can be analysed similarly. Replacement of the characteristic polynomials by

$$\rho(\zeta) = -\frac{1}{11}(4\zeta^2 + 7\zeta), \quad \sigma(\zeta) = \frac{1}{11}(28\zeta^2 - 10\zeta)$$

implies the two conditions

$$P(x,y) = 1 - \frac{139}{132}x + \frac{59}{121}x^2 + \frac{23}{363}y^2 + \frac{49}{4356}x^4 + \frac{49}{2178}x^2y^2 + \frac{49}{4356}y^4 - \frac{973}{8712}x^3 - \frac{973}{8712}xy^2 > 0 \tag{40}$$

and

$$\begin{aligned} Q(x,y) = &\frac{931}{2108304}x^2y^6 + \frac{931}{1405536}x^4y^4 + \frac{931}{2108304}x^6y^2 \\ &+ \frac{125327}{383328}x^2y^2 - 2x - \frac{29947}{1054152}xy^4 - \frac{35311}{234256}x^3y^2 \\ &- \frac{571}{2904}xy^2 + \frac{157}{66}x^2 - \frac{19}{726}y^4 - \frac{3869}{2904}x^3 \\ &+ \frac{179711}{383328}x^4 - \frac{139}{351384}y^6 - \frac{581}{209088}x^7 - \frac{257905}{2108304}x^5 \\ &+ \frac{931}{8433216}y^8 + \frac{931}{8433216}x^8 + \frac{2382433}{101198592}x^6 \\ &- \frac{581}{209088}xy^6 - \frac{581}{69696}x^3y^4 - \frac{581}{69696}x^5y^2 \\ &+ \frac{2302369}{101198592}x^2y^4 + \frac{2362417}{50599296}x^4y^2 > 0 \end{aligned} \tag{41}$$

Again with a help of *Maple* we draw the region of stability (Fig. 2) and can conclude that the scheme of Adams type-B is $A(\theta)$-stable.

6. Concluding remarks

The present note provides two pairs of Adams-type two-step LALMM of fourth-order convergence. Their stability analysis gives $A(\theta)$-stability. These facts suggest their applicability to practical problems. Some preliminary numerical experiments showed their competetiveness with conventional DVMs. However, several issues are remained to challenge.

- Full numerical experiments, in particular with system of ODEs
- Development and analysis of Milne-type schemes, mentioned in §2.
- Exploration of *a posteriori* error estimation of the schemes and its application to an automatic stepsize control strategy

They will be reported in succeeding notes.

Acknowledgements. The first author is supported by the Grants-in-Aid for Scientific Research (No.

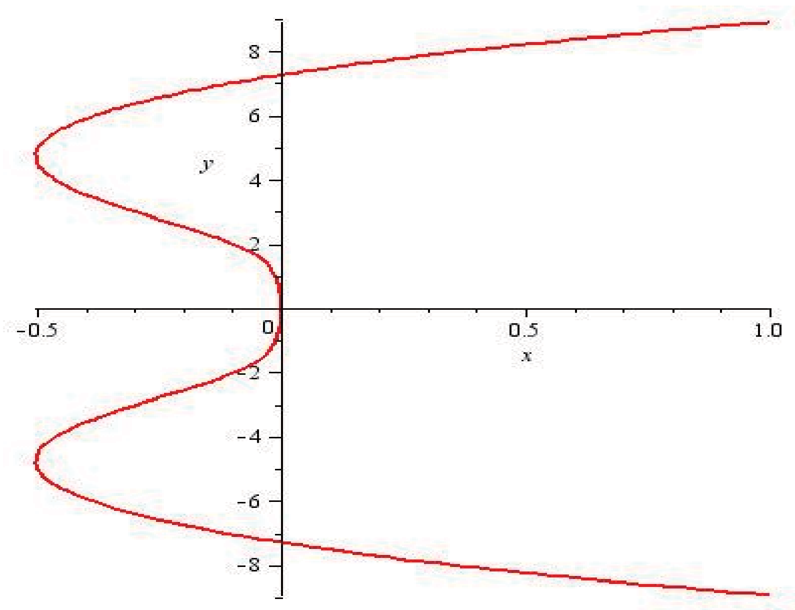

Fig. 1. Region of stability of LALMM type-A (Left-hand side of the contour line).

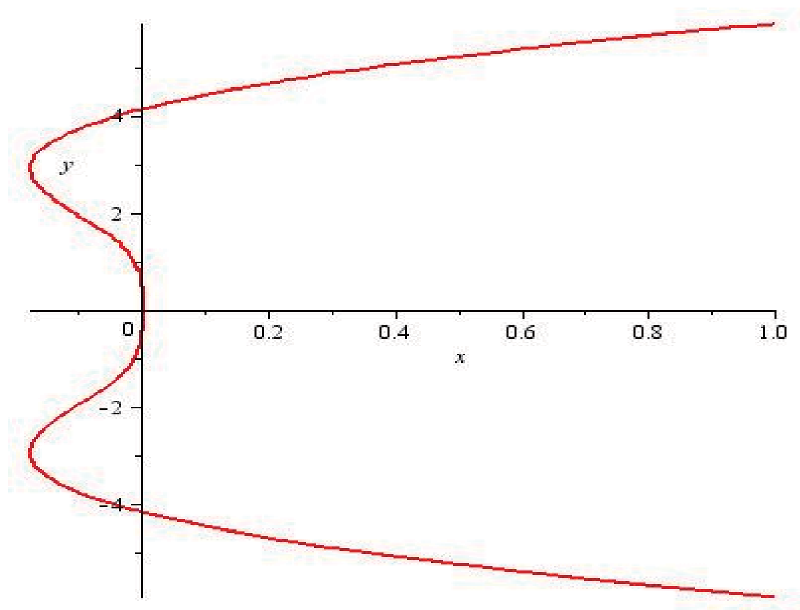

Fig. 2. Region of stability of LALMM type-B (Left-hand side of the contour line).

21540153) supplied by Japan Society for the Promotion of Science.

References

1) E. Hairer, S. P. Nørsett, and G. Wanner, Solving Ordinary Differential Equations I, Nonstiff Problems, Second Revised Edition, Springer-V., Berlin, 1993.

2) T. Mitsui, "'Look-ahead" linear multistep methods for ordinary differential equations — Introduction of the method —", Science and Engineering Review of Doshisha Univ., 51:181–190, 2010.

3) R. A. Usmani and R. P. Agarwal, "An A-stable extended trapezoidal rule for the integration of ordinary differential equations", Comput. & Math. with Appls., 11:1183–1191, 1985.

4) I. B. Jacques, "Extended one-step methods for the numerical solutions of ordinary differential equations", Intern. J. Computer Math., 29:247–255, 1989.

5) Y. Inamasu, Y. Kaneko, and H. Yanagiwara, "On a multistep method", J. Fac. International Studies of Culture, Kyushu Sangyou Univ., (1):179–196, 1994.

6) M. Urabe, "An implicit one-step method of high-order accuracy for the numerical integration of ordinary differential equations", Numer. Math., 15:151–164, 1970.