# Self-Organizing Small-World Structure of Neural Networks by STDP Learning Rule

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Spike-timing-dependent plasticity (STDP) learning strengthens or weakens synaptic weights of a neural network, thus the neural network temporally evolves by the STDP rule. By estimating the characteristic path length and clustering coefficient, we examined how the neural network structure changes and synaptic spikes synchronize. Even if the neural networks do not have any initial structure, small-world characteristics emerge; the characteristic path length is as small as that of a random graph, but the clustering coefficient is greater.

Key words: Neural-network model, Spike-timing-dependent plasticity (STDP), Small-world networks

## 1. Introduction

The performance of transmitting spike information between neurons changes temporally. This synaptic plasticity is considered to be an essential property of learning and memorizing information. Recent experimental studies have revealed a new learning rule for a neural network; this rule strengthens or weakens synaptic weights in the neural network according to the spike timing between pre-synaptic and post-synaptic firings. The rule is termed spike-timing-dependent plasticity (STDP) and has been observed in several real neural networks [1–6].

The STDP rule enables the neural network to temporally modify its structure to strengthen the synchrony of neuron firings, there by memorizing a large amount of information more accurately. Temporal changes in the distribution of synaptic weights are mainly discussed by employing the Fokker-Plank equation [7]. Although these studies [7] often reveal some statistical aspects of synaptic distributions, we have to consider another important aspect—the network structure. One of the reasons is that neurons can be connected to each other by short path length and they are highly clustered due to STDP. Such a structure cannot be characterized only by the distribution of the synaptic weights.

An interesting concept in complex networks is the small-world network model proposed by Watts and Strogatz, which is more highly clustered than a random graph yet has as small characteristic path length as that of the random graph [8, 9]. For example, they showed that a neural structure of *C. elegance* exhibits small-world network characteristics. It was also reported that the small-world route achieves synchronization more efficiently than a pure random graph [10]. These results indicate that the STDP learning rule can transform a neural network structure into a small-world structure by dynamically modifying the synaptic weights even when the initial network is a random graph.

In this paper, we used an artificial neural network along with the Nagumo-Sato neuron model [11–13] to

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examine how the network structure changes and how synaptic spikes synchronize by evaluating the characteristic path length and clustering coefficient [8, 9]. In other words, we analyzed the dynamical change in the network structure, that is, the spatial structure of the synaptic weights. The reason for employing the Nagumo-Sato neuron model is that it has been studied as a recurrent neural network model [13] and is considered to be an origin of chaotic neural networks [14, 15]. However, we believe that qualitatively similar results would be shown for different types of neurons, for example, a leaky integrate-and-fire neuron In addition, the case of the FitzHugh-Nagumo neuron model has been reported by Ref. (16).

## 2.   Models

### 2.1 Neural network model

To investigate a dynamical change in the network structure, we constructed a partially connected neural network with the Nagumo-Sato neuron model [12, 13] as follows:

$$
\begin{aligned}
y_i(t+1) = & \sum_{j=1}^{N} \sum_{d=0}^{t} \exp\left(\frac{-d}{\tau_m}\right) s_j w_{ij} x_j(t-d) \\
& + \sum_{d=0}^{t} \exp\left(\frac{-d}{\tau_s}\right) I_i(t-d) \\
& - \alpha \sum_{d=0}^{t} \exp\left(\frac{-d}{\tau_r}\right) x_i(t-d) - \theta
\end{aligned}
\tag{1}
$$

where $y_i(t)$ is the internal state of the $i$-th neuron at time $t$, $N$ is the number of neurons ($N = 225$); $x_i(t)$ is the output of the $i$-th neuron, $s_j$ decides whether the $j$-th neuron is excitatory or inhibitory ($s_j = 1$ or $s_j = -1$, respectively)—approximately 80% of neurons are excitatory, and $w_{ij}$ is the synaptic weight from the $j$-th to the $i$-th neuron. Initial synaptic weights are uniformly distributed in the range of [0.9, 1.1], as shown in the leftmost part of Fig. 3(c); the percentage of connected weights ($w_{ij} > 0$) is approximately 20% excluding self-feedback connections ($w_{ii} = 0$). In Eq. (1), the first term integrates inputs from pre-synaptic neurons, the second term integrates external inputs, and the third term expresses the refractoriness of a neuron firing. If the sum of the three terms is greater than the threshold $\theta$, the $j$-th neuron emits a spike. The Heaviside function is the activation function between the internal state $y_j(t)$ and output $x_j(t)$.

As external inputs, spatiotemporal patterns were produced by a Poisson process in which the probability of an input occurrence is 0.05, and $I_i = 5$ if the input exists, otherwise $I_i = 0$. The temporal epoch of the pattern is $T$. The spatiotemporal patterns were applied repeatedly to the neural network, as shown in Fig. 1.
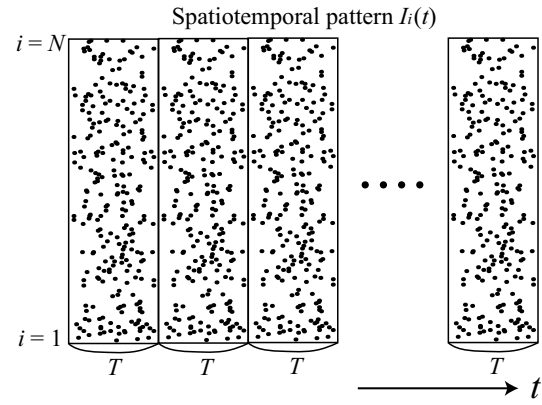


Fig. 1. Spatiotemporal patterns are repeatedly applied as external information learned by the neural network. The period of the pattern is $T$, and each dot represents an input $I_i(t)$ for each neuron. The spatiotemporal patterns were produced by a Poisson process.

### 2.2 STDP learning rule

The STDP rule [5] for modifying the synaptic weights $w_{ij}$ is expressed below:

$$
\Delta w_{ij}(\Delta t_{ij}) = 
\begin{cases}
A e^{\frac{\Delta t_{ij}}{\tau_p}} & (\Delta t_{ij} > 0) \\
-A e^{-\frac{\Delta t_{ij}}{\tau_n}} & (\Delta t_{ij} \leq 0)
\end{cases}
$$

where $A$ is a parameter that controls the maximum plasticity modification ($A = 0.1$) and $\Delta t_{ij}$ is the temporal difference between a pre-synaptic event and a post-synaptic spike, that is, $\Delta t_{ij} = t_j - t_i$. The STDP rule was applied only to excitatory-to-excitatory connections, and other connections were fixed. We used different time constants for potentiation and depression: $\tau_p = 10$

and $\tau_n = 20$. Moreover, the synaptic weights were limited to the range of $[0, 2]$.

## 3. Dynamical change of the network by STDP

### 3.1 Characterizing a temporal change of a network structure

We define two measures to examine the temporal dynamics of a neural network learned by the STDP rule. The first measure evaluates the degree of synchronization of neuron firings, the synchrony coefficient, and it is defined as follows:

$$s(t) = \frac{2}{N(N-1)\mu} x_i(t-d) x_j(t-d)$$

where $\quad \mu = \frac{1}{N} \sum_{i=1}^{N} \sum_{d=T-1}^{0} x_i(t-d)$ .

The second measure evaluates the network structure from the viewpoint of the small-world network; we introduced the characteristic path length $L$ and the clustering coefficient $C$ [8, 9]. The characteristic path length $L$ is defined as the average of the shortest path lengths $d_{ij}$ between two vertices, which represents the minimum number of edges traversed from a vertex $i$ to another vertex $j$ :

$$L(t) = \frac{1}{N(N-1)} \sum_{i \neq j \in \mathbf{G}(t)} d_{ij}(t)$$

where $\mathbf{G}(t)$ represents the graph (the neural network structure) at time $t$. The clustering coefficient $C$ is defined as the average of local clustering of the $i$-th node (the neuron):

$$C_i(t) = \frac{\text{Number of edged in } \mathbf{G}_i(t)}{{}_{k_i}\mathbf{C}_2}$$

where $\mathbf{G}_i(t)$ is a subgraph of nearest neighbors of node $i$ and $k_i$ is the number of nearest neighbors. Then, the clustering coefficient $C$ is

$$C(t) = \frac{1}{N} \sum_{i \in \mathbf{G}_i(t)} C_i(t)$$

In this paper, we define the degree of a small-world structure as follows:

$$S(t) = \frac{C(t)/C_r(t)}{L(t)/L_r(t)} \quad (2)$$

where $C_r(t)$ is the clustering coefficient of a perfectly random rewired network whose synaptic weight distribution is the same as that of the network used for calculating $C(t)$, and $L_r(t)$ is the clustering coefficient of the random rewired network. In addition, only excitatory-to-excitatory connections were used for calculating $C(t)$, $C_r(t)$, $L(t)$, $L_r(t)$, and $S(t)$. If a network has a small-world structure, $S(t)$ becomes greater than 1 because this structure is more highly clustered than a random network. Although Eq. (2) is useful for evaluating the structure, we still have to consider an important difference between the neural network and the undirectional and unweighted graph analyzed in Ref. (8, 9): the synaptic weights are usually analog values. Therefore, we should essentially evaluate the small-world index (Eq. (2)) for a directed and weighted graph.

However, because there is no accepted measure of the clustering coefficient for a directed and weighted connection, we simplified the network connections. We reduced the neural network structure into the simplest one, as shown in Fig. 2, without losing the essence of a temporal change in the neural network structure.
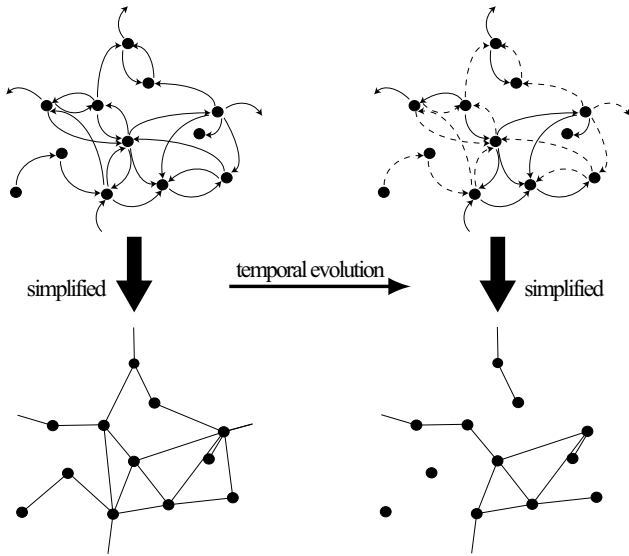
FIG. 2: Simplification of connections of the neural network into an undirected and unweighted graph for characterizing the network structure. For this simplification, weakened connections (dashed arrows) after a temporal evolution were considered to disappear if $w_{ij}$ <0.5.



Fig. 3. Simulation results when $N$=225, $\tau_m$=$\tau_s$=5, $\tau_r$=3, $\theta$=4, $T$=100, $A$=0.1, $\tau_p$=10, and $\tau_n$=20. (a) Raster plot, (b) degree of synchronization, (c) temporal change in the synaptic weight distributions by the STDP learning rule, which is applied only to excitatory-to-excitatory connections, and (d) the small-world index. As $S(t)$ exceeds 1, the small-world structure appears more clearly.

*3.2 Simulation results*

By employing the numerical techniques described above, we investigated how the network structure learned by the STDP rule temporally changes and how the synaptic spikes synchronize. The simulation results are shown in Fig. 3.

Figures 3(a) and 3(b) show the response due to the repeatedly applied spatiotemporal patterns shown in Fig. 1. Synchronous firing gradually emerges as the learning proceeds. Figure 3(c) shows the temporal evolution of the synaptic weight distribution. As time evolves, the synaptic weights are polarized; they are either strengthened or weakened. Because the condition for maintaining the connection of $w_{ij}$ is $w_{ij} \geq 0.5$, the number of weakened connections reduces gradually. Thus, only those connections essential for learning are retained and they are strengthened. In other words, the structure of the neural network is optimized for coding input information. Moreover, we confirmed that the optimized network is transformed from an initial random network to a small-world network whose value of $S$ becomes greater than 1, as shown in Fig. 3(d). That
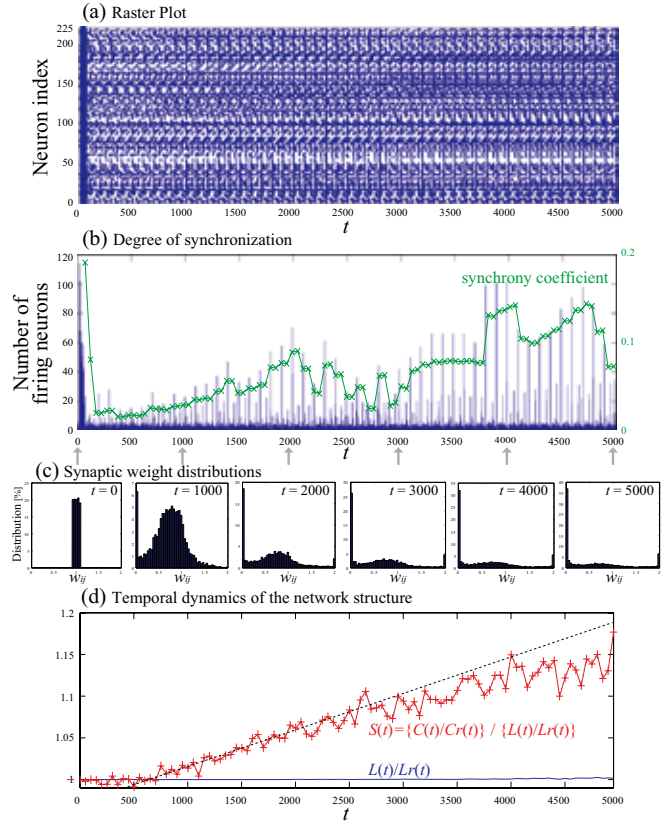
is, the characteristic path length is as small as that of a random graph, but the clustering coefficient is greater. The important point is that although none of the initial conditions of the neural network have any order, that is, the network has a random structure and the input spatiotemporal patterns are random, the order of the structure is characterized by the small-world network after the learning.

## 4.   Conclusions

We focused on the temporal dynamical change of a neural network structure learned by the STDP rule. The contributions of this paper are summarized as follows. First, we exhibited the enhancement in the synchronization of neurons when the synaptic weights

were polarized temporally. The polarization may optimize the network structure such that it can learn external information efficiently. Next, we investigated how such a network changes from the viewpoint of a small-world network. We confirmed that through the STDP learning rule, the network is autonomously transformed from an initial random network to a small-world network, that is, an optimized network.

In future study, it is important to generalize measures for characterizing the network structures of directed, weighted, and multiple connections.

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