Studies on Mathematical Structures of Network Optimization Problems

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September 2013

Contents

1 Introduction

The purpose of the present thesis is to investigate the mathematical structures of combinatorial optimization problems on flow-networks from various view points. The combinatorial optimization problems on flow-networks form the most typical and the most important class in vast field of combinatorial optimization problems. Lots of problems in engineering and social science can be described by the terminology of graph theory, so the optimization problems on flow-networks arises in the various fields of science and engineering.

The main results of the present thesis are divided into two parts. The first part deals with the maximum flow problem. We formulate the maximal flow problem as the LP or the IP problem and discuss the duality between the flows and the cutset in terms of the LP formulation of the maximum flow problem. Further, we determine the universal Gröbner basis associated with the maximum flow problem. The second part consists of the results on the eigenvalue problem of matrices with values in Min-Plus algebra. Although the first part and the second part of the thesis are both concerned with networks on graphs, it seems that they are mutually independent.

In the first part of the present thesis, we deals with the maximum flow problem formulated as the linear programming (LP) problem or the integer programming (IP) problems. The LP problem is the problem of minimizing (or maximizing) a linear form subject to the constraints consisting of linear equalities and inequalities; if the values of the variables in the LP problem are restricted to integers, the LP problem is called the IP problem. If an optimization problem on the flow-network is formulated as the LP or the IP problem, it can be solved by the general algorithms for the LP or the IP problem. However, for the many of the famous optimization problems on flow-networks, efficient and specified combinatorial algorithms have been developed. Such algorithms solve the problems much faster than the general LP or IP algorithms. Then what is the significance of the formulation of the combinatorial optimization problems as the LP or the IP problems? One of the major advantages of the formulation of the combinatorial optimization problem as the LP or the IP problem is the flexibility for the change of constraints. We will pay a little attention to such practical advantages of the LP or the IP formulation. We will take notice of the other advantage of the formulation. Once the combinatorial optimization problems is formulated as the LP or the IP problem, one can use the technique from the algebra or geometry in order to analyze the structure of the problems. In the present thesis, we investigate the structures of various kinds of optimization problems on the flow-networks using various tools from algebra or geometry. In such investigation, the formulation of problems on flow-networks as the LP or the IP problems play an important role.

In the first part of the present thesis, we focus on the maximum flow problem which is a well-known optimization problem on flow-networks. The maximum flow problem is the problem for finding the flow such that the flow value is maximal among the flows subject to the capacity restriction and the flow conservation laws. There are various combinatorial algorithms for solving the maximum flow problems as follows: The augmenting path algorithm due to Ford and Fulkerson [11] is the first one and most popular; the preflow-push algorithm due to Goldberg and Tarjan [14] is the latest and probably the most efficient one. The max-flow min-cutset theorem which is one of the most famous results in the theory of flow-network problem suggests the duality between flows and cutsets. In the present thesis we give a formulation of the maximum flow problem as an LP problem and try to explain the duality between the flows and cutsets from the view point of LP duality. We have seen only by computational experiments that the LP dual for the maximum flow problem returns the binary vector expressing the min-cut and min-cutset as the optimal solution but we have not yet obtained the rigorous proof of the result. Especially we do not know the reason why the LP dual of the maximum flow problem returns the binary vector as the optimal solution. Next we clarify the mathematical structure of the maximum flow problem in some sense through the Gröbner basis associated with the problem. The notion of the Gröbner basis together with the algorithm for its construction were introduced by Buchberger [5] in 1960's in order to solve various problems in the polynomial ideal theory [6]. Since then lots of algorithms based on the Gröbner basis have been exploited for solving various kinds of problems in the polynomial ideal theory and implemented to computer algebra systems [6]. Therefore the Gröbner basis becomes indispensable not only to the polynomial ideal theory but also to the development of computer algebra. We are mainly interested in the application of Gröbner basis technique to combinatorial optimization problems [21, 29]. In the application to combinatorial optimization problems, the key technique is Conti-Traverso's algorithm [6, 7, 22] for solving integer programming (IP) problems. In the algorithm, Gröbner basis of toric ideals play a crucial role. In order to do such investigation, we give another formulation of the maximum flow problem as the LP problem slightly different from the above LP formulation. Moreover, we have given the characterization of the universal Gröbner basis associated with the maximum flow problems by combinatorial property of the graph. We have proved that the universal

Gröbner basis of toric ideal associated with our formulation of the maximum flow problem as the IP problems consists of binomials corresponding to all incidence vectors of circuits and s-t paths of the digraph. Our result directly follows from the well known results on toric ideals associated with incidence matrices of digraphs in combination with the properties of "Lawrence lifting". In addition to the main result, we have obtained some results on the toric ideal associated with the reduced incidence matrices. We prove that the kernel lattice of the reduced incidence matrices are generated by incidence vectors of s-t directed paths under some assumption for the digraph; by using this observation , we determine the generators of the toric ideal associated with reduced incidence matrices of digraphs. It is known that the computation of Gröbner basis by using Buchberger's algorithm takes a tremendously long time and consumes a huge amount of storage space. In fact, computational experiments show that computation of the Gröbner basis of toric ideals associated with the maximum flow problem takes too much time even for the problems with particularly small size [21]. In order to verify the efficiency of our results in computing Gröbner basis of toric ideals associated with maximum flow problem, we have to develop the efficient algorithm for enumerating paths and circuits of digraphs and implement it to a suitable computer algebra system. However, we do not know the efficient algorithm for such enumeration. We will remain such kind of investigation for the future study.

In the second part of the present thesis, we focus on the eigenvalue problem of matrices with entries in min-plus algebra. Min-Plus algebra is the set of all real numbers and the element the infinity with the binary operations "min" and "+". It is one of many idempotent semirings, and has been studied in various fields of mathematics. Many of theorems and techniques that we use in usual linear algebra seems to have analogues in linear algebra over min-plus algebra. Moreover, Min-Plus algebra may seems to fit well with the algorithm of the network optimization problems. However, such kind of investigation have not yet exploited sufficiently. In the present thesis, we consider the eigenvalue problem of matrices with entries in min-plus algebra and characterize the eigenvalues and corresponding eigenvectors using the terminology of the network theory on digraphs. First we define the network associated with the matrix with entries in min-plus algebra. Then, we show that the minimal average weight of the circuits in the network become the eigenvalue of the matrix. Also we show that the corresponding eigenvectors appears as the column vectors of the minimal weight matrix of the specified network which is obtained from the original network by subtracting the minimal average weights from the weight of every edges. Further, we prove under some assumption that the minimal average weight of the network is the unique eigenvalue of the matrix. Finally we refer to the coincidence between the right eigenvalue and the left eigenvalue.

The present thesis is organized as follows: In section 2, we give a brief review of graph theory, and introduce some of the network optimization problem. In section 3, we give the definition of the LP problem, especially the definition of the standard form LP problem and the IP problem and we also give the definition of the dual problem. In section 4, we formulate some of the network optimization problem as the standard form LP problem. We explain the relation between the maxflow and the mincutset through the LP duality. In section 5, we give some basic notations and terminologies for polynomial ideal theory and give the definition of the Gröbner basis. We prove the first main result asserting that the universal Gröbner basis of toric ideal associated with the maximum flow problem consists of binomials corresponding to the incidence vectors of all circuits and s-t paths of the digraph. In section 6, we discuss the eigenvalue problem of matrices with entries in Min-Plus algebra. We determine the eigenvalues and the eigenvectors belonging to the eigenvalues. The second main result of the thesis is described in terms of the terminology of the network theory on graphs.

2 Network Flows

2.1 Digraphs

A directed graph or, for short, a digraph G consists of the finite sets V and E; an element $v \in V$ is called a vertex and an element $e \in E$ is called an edge of G. The edge $e \in E$ can be expressed as an ordered pair $e = (v_i, v_j)$ of vertices $v_i, v_j \in V$. We introdece the maps $\partial^- : E \to V$ and $\partial^+ : E \to V$ by $\partial^-(e) = v_i$, $\partial^+(e) = v_j$ for $e = (v_i, v_j)$; vertices v_i and v_j are called the tail and the head of the edge $e = (v_i, v_j)$ respectively; vertices v_i and v_j are simply called the end-vertices of $e = (v_i, v_j)$. If distinct edges e and e' have two end-vertices in common, then one of the cases (i) $\partial^-(e) = \partial^-(e')$ and $\partial^+(e) = \partial^+(e')$ or (ii) $\partial^-(e) = \partial^+(e')$ and $\partial^+(e) = \partial^-(e')$ occurs; in the former case, edges e and e' are called parallel edges and latter case, they are called antiparallel edges. An edge with just one end vertex, that is, $\partial^-(e) = \partial^+(e)$ holds is called a loop. A graph without loops, parallel edges and antiparallel edges is called simple. A sequence of vertices $W = (v_{i_0}, v_{i_1}, \ldots, v_{i_l})$ in G is called a walk if $(v_{i_{k-1}}, v_{i_k}) \in E$ or $(v_{i_k}, v_{i_{k-1}}) \in E$ for $k = 1, \ldots, l$. Vertices v_{i_0} and v_{i_l} are respectively called an initial vertex and a terminal vertex of the walk W. Edges $e_{i_k} = (v_{i_{k-1}}, v_{i_k})$ or $e_{i_k} = (v_{i_k}, v_{i_{k-1}})$ in the walk are called a forward edge or a backward edge according to whether $e_{i_k} = (v_{i_{k-1}}, v_{i_k})$ or $e_{i_k} = (v_{i_k}, v_{i_{k-1}})$. A walk W is called directed if all the edges in W are forward edges. A walk T is called a trail if the edges in T are pairwise distinct. A walk P is called a path if the vertices in P are pairwise distinct except the initial vertex and the terminal vertex. A path with the initial vertex v_{i_0} and with the terminal vertex v_{i_l} is called a v_{i_0}-v_{i_l} path. If the initial vertex and the terminal vertex of the path is identical, then the path is called closed; a closed path is called a circuit. Henceforth, we sometimes express walks, paths and circuits in terms of the sequence of edges. For example, we write the walk $(v_{i_0}, v_{i_1}, \ldots, v_{i_l})$ with $e_{i_k} = (v_{i_{k-1}}, v_{i_k})$ or $(v_{i_k}, v_{i_{k-1}})$ as $W = (e_{i_1}, e_{i_2}, \ldots, e_{i_l})$. We express directed walks, directed paths and directed circuits in terms of the alternating sequence of vertices and edges. For the directed walk $(v_{i_0}, v_{i_1}, \ldots, v_{i_l})$ with $e_{i_k} = (v_{i_{k-1}}, v_{i_k})$, we write $W = (v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_l}, v_{i_l})$.

Definition 2.1. Let $E = \{e_1, \ldots, e_m\}$ be a set of edges in a digraph G.
(1) Let $\mathcal{W} = (e_{i_1}, \ldots, e_{i_l})$ be a directed walk expressed in terms of the sequence of edges. We define the incidence vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ of $\mathcal{W}$ as follows:

(i) If $e_{i_k} \in E$ is included j times in the directed walk $\mathcal{W}$, then we set $\omega_{i_k} = j$.

(ii) If $e_{i_k} \in E$ is not included in the directed walk $\mathcal{W}$, then we set $\omega_{i_k} = 0$.

(2) Let $P = (e_{i_1}, \ldots, e_{i_l})$ be a path expressed in terms of the sequence of edges. Then edges in the path P is divided into the disjoint union of a set of forward edges P^+ and a set of backward edges P^-. We define the incidence vector $\boldsymbol{p} = (p_1, \ldots, p_m)$ of a path P by

$$p_k = \begin{cases} +1 & \text{if} \quad e_k \in P^+ \\ -1 & \text{if} \quad e_k \in P^- \\ 0 & \text{if} \quad e_k \notin P \; . \end{cases}$$

In particular, if all entries of the incidence vector $\boldsymbol{p}$ of the path P are non-negative, then the path P is called directed.

Definition 2.2. Let $G = (V, E)$ be a digraph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$.
(1) We define the adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ of G by

$$a_{ij} = \begin{cases} 1 & \text{if} \quad (v_i, v_j) \in E \\ 0 & \text{if} \quad (v_i, v_j) \notin E \end{cases} \; .$$

(2) We define the incidence matrix $Q = (q_{ik}) \in \mathbb{R}^{n \times m}$ of G by

$$q_{ik} = \begin{cases} +1 & \text{if } \partial^+(e_k) = v_i \\ -1 & \text{if } \partial^-(e_k) = v_i \\ 0 & \text{otherwise} \; . \end{cases}$$

2.2 Minimum Cost Flow Problem

Let $G = (V, E)$ be a digraph with n vertices and m edges. We assign a positive integer $c(e)$ to each edge $e \in E$; $c(e)$ is called the capacity of the edge e. We also assign a positive integer $\delta(e)$ to each edge $e \in E$ in addition to the capacity; $\delta(e)$ is called the cost of the edge e. Moreover, we assign the integer $d(v)$ to each vertex $v \in V$ with $\sum_{v \in V} d(v) = 0$; $d(v)$ is called the demand. The quadruple $\mathcal{N} = (G, c, \delta, d)$ is called a network on G associated with the minimum cost flow problem. A flow on the network $\mathcal{N}$ is the function f on E satisfying the following conditions (i) and (ii):

(i) The capacity constraint $0 \le u_k := f(e_k) \le c_k := c(e_k)$ for all $k = 1, \ldots, m$.

(ii) The demand condition at each vertex:

$$\sum_{\partial^+(e_k) = v_i} u_k - \sum_{\partial^-(e_k) = v_i} u_k = d(v_i) \quad (v_i \in V) \; .$$

The minimum cost flow problem is the problem for finding the flow $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$ such that the cost $\sum_{e_k \in E} \delta(e_k) u_k$ is minimal. That is, the minimum cost flow problem is formulated as follows:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{e_k \in E} \delta(e_k) u_k \\ \text{subject to} & \displaystyle\sum_{\partial^+(e_k) = v_i} u_k - \sum_{\partial^-(e_k) = v_i} u_k = d(v_i) \quad (v_i \in V) \\ & 0 \leq u_k \leq c_k \quad (k = 1, 2, \cdots, m) \end{array}$$

2.3 Shortest Path Problem

Let $G = (V, E)$ be a digraph with the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and the set of edges $E = \{e_1, e_2, \ldots, e_m\}$. We assign a positive integer $w(e)$ to each edge $e \in E$; $w(e)$ is called the weight of the edge e. Moreover, we specify the vertices $s = v_1$ (the source) and $t = v_m$ (the sink). The quadruple $\mathcal{N} = (G, w, s, t)$ is called a network on G associated with the shortest path problem.

Definition 2.3. Let $\mathcal{N}$ be a network associated with the shortest path problem and let $P = (e_{i_1}, e_{i_2}, \ldots, e_{i_l})$ be a directed path. The length $\ell(P)$ of the path P is the number l of edges in P; the weight $\sigma(P)$ of the path P is the sum of weights of edges in P:

$$\sigma(P) = \sum_{k=1}^{l} w(e_{i_k}) \ .$$

For a circuit C, we define the length $\ell(C)$ and the weight $\sigma(C)$ of C in the same way as for paths.

The shortest path problem is the problem for finding the s-t directed path whose weight is minimal in the network $\mathcal{N}$.

2.4 Maximum Flow Problem

Let $G = (V, E)$ be a digraph with the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and the set of edges $E = \{e_1, e_2, \ldots, e_m\}$. The digraph G has the source $s = v_1$ and the sink $t = v_m$, and each edge $e \in E$ has the capacity $c(e)$; the quadruple $\mathcal{N} = (G, c, s, t)$ is called a network associated with the maximum flow problem. A flow on the network $\mathcal{N}$ is the function f on E satisfying the following conditions (i) and (ii):

(i) The capacity constraint $0 \leq u_k := f(e_k) \leq c_k := c(e_k)$ for all $k = 1, \ldots, m$.

(ii) The flow conservation law at each vertex except for the source s and the sink t:

$$\sum_{\partial^+(e_k)=v_i} u_k = \sum_{\partial^-(e_k)=v_i} u_k \quad (v_i \in V \backslash \{s,t\}) \ .$$

It follows from the flow conservation law that we have

$$\tau(\boldsymbol{u}) = \sum_{\partial^-(e_k)=s} u_k = \sum_{\partial^+(e_k)=t} u_k \ .$$

$\tau(\boldsymbol{u})$ is called the flow value of the flow $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$. The maximum flow problem is the problem for finding the flow such that the flow value is maximal. That is, the maximum flow problem is formulated as follows:

$$\begin{array}{rl} \text{maximize} & \tau(\boldsymbol{u}) = \displaystyle\sum_{\partial^-(e_k)=s} u_k = \sum_{\partial^+(e_k)=t} u_k \\ \text{subject to} & \displaystyle\sum_{\partial^+(e_k)=v_i} u_k - \sum_{\partial^-(e_k)=v_i} u_k = 0 \quad (v_i \in V \backslash \{s,t\}) \\ & 0 \leq u_k \leq c_k \quad (j = 1, 2, \cdots, m) \end{array}$$

We explain the Maxflow-Mincutset theorem which is well-known theorem that describe the duality in the flow network. If a vertex set V is divided into the disjoint union $V = \Phi \sqcup (V \setminus \Phi)$ and $s \in \Phi$, $t \in V \setminus \Phi$, then the set Φ is called a cut. The set of edges Ψ whose elements e_k satisfy $\partial^-(e_k) \in \Phi$ and $\partial^+(e_k) \in V \setminus \Phi$ is called the cutset of the cut Φ.

Definition 2.4.
(1) For a cut $\Phi \subset V = \{v_1, \ldots, v_n\}$ of a digraph $G = (V, E)$, we define the incidence vector $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$ of Φ by

$$\phi_i = \begin{cases} 1 & v_i \in \Phi \\ 0 & v_i \in V \setminus \Phi \ . \end{cases}$$

(2) For a cutset $\Psi \subset E = \{e_1, \ldots, e_m\}$ of a digraph $G = (V, E)$, we define the incidence vector $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_m)$ of Ψ by

$$\psi_i = \begin{cases} 1 & e_i \in \Psi \\ 0 & e_i \in E \setminus \Psi \ . \end{cases}$$

Let $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_m)$ be the incidence vector of a cutset Ψ. Then the capacity $\sigma(\Psi)$ of Ψ is defined as

$$\sigma(\Psi) = \sum_{k=1}^{m} \psi_k c(e_k) \ .$$

If the capacity $\sigma(\Psi)$ of the cutset Ψ satisfies the inequality $\sigma(\Psi) \leq \sigma(\Psi')$ for the capacity $\sigma(\Psi')$ of an arbitrary cutset Ψ', then the cutset Ψ is called minimal.

Theorem 2.5 (Maxflow-Mincutset Theorem, [1, 12, 19])**.**
In the network $\mathcal{N}$ associated with the maximium flow problem, the maximam flow value in $\mathcal{N}$ is equal to the capacity of the minimal cutset in $\mathcal{N}$.

3 LP Problem and IP Problem

3.1 The Standard Form of LP and IP

Definition 3.1 (LP Problem and IP Problem)**.**
(1) Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix with entries in $\mathbb{R}$ and let $\boldsymbol{b} \in \mathbb{R}^m$ $\boldsymbol{c} \in \mathbb{R}^n$ be column vectors. The linear programming (LP) problem in the standard form is the problem of finding the column vector $\boldsymbol{x} \in \mathbb{R}^n_{\geq 0}$ with non-negative entries satisfying the condition $A\boldsymbol{x} = \boldsymbol{b}$ and minimizes the linear form ${}^t\boldsymbol{c}\boldsymbol{x} = c_1x_1 + \cdots + c_nx_n$. We write this LP problem simply as:

$$\begin{aligned} \text{minimize} \quad & {}^t\boldsymbol{c}\boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in \mathbb{R}^n_{\geq 0} \ . \end{aligned} \tag{3.1}$$

(2) Let $A = (a_{ij}) \in \mathbb{Z}^{m \times n}$ be an $m \times n$ matrix, $\boldsymbol{b} \in \mathbb{Z}^m$ be a column vector with integer entries and $\boldsymbol{c} \in \mathbb{R}^n$ be a column vector with real entries. The integer programing (IP) problem in the standard form is the problem of finding the column vector $\boldsymbol{x} \in \mathbb{Z}^n_{\geq 0}$ with non-negative integer-entries satisfying $A\boldsymbol{x} = \boldsymbol{b}$ and minimize ${}^t\boldsymbol{c}\boldsymbol{x}$. We also write this IP problem simply as:

$$\begin{aligned} \text{minimize} \quad & {}^t\boldsymbol{c}\boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in \mathbb{Z}^n_{\geq 0} \ . \end{aligned} \tag{3.2}$$

The vector $\boldsymbol{x}$ subject to the condition is called the feasible solution. If a value of the objective function ${}^t\boldsymbol{c}\boldsymbol{x}$ which is obtained by the feasible solution $\boldsymbol{x}$ is minimal, then the feasible solution $\boldsymbol{x}$ is called the optimal solution. The minimal value ${}^t\boldsymbol{c}\boldsymbol{x}$ is called the optimal value.

3.2 Dual Problem

Definition 3.2 (Dual Problem)**.**
For the LP problem formulated as (3.1), its dual problem is writen as

$$\begin{array}{rl} \text{maximize} & {}^t\boldsymbol{b}\boldsymbol{y} \\ \text{subject to} & {}^tA\boldsymbol{y} \le \boldsymbol{c},\ \boldsymbol{y} \in \mathbb{R}^m\ . \end{array} \tag{3.3}$$

If we consider the objective function of (3.1) from minimization to maximization, we can write its dual problem as follows:

$$\begin{array}{rl} \text{minimize} & {}^t\boldsymbol{b}\boldsymbol{y} \\ \text{subject to} & {}^tA\boldsymbol{y} \ge \boldsymbol{c},\ \boldsymbol{y} \in \mathbb{R}^m\ . \end{array} \tag{3.4}$$

The following theorem shows the relation between the LP problem and its dual problem.

Theorem 3.3 (Duality Theorem, [2, 13])**.**
We consider the LP problem in the standard form (3.1) and its dual problem (3.3). If either the LP problem or its dual problem has the optimal solution, then the other also has a optimal solution and their optimal value coinside.

4 Network Flows and LP Problem

4.1 Minimum Cost Flow and Shortest Path as LP

First, we show the formulation of the minimum cost flow problem as the LP problem. Let $\mathcal{N} = (G, c, \delta, d)$ be a network associated with the minimum cost flow problem on the digraph G with n vertices and m edges, and let $Q \in \mathbb{R}^{n\times m}$ be the incidence matrix of G. In order to formulate the minimum cost flow problem as the LP problem in the standard form (3.1), we introduce the vector $\boldsymbol{c}$ whose k^{th} entry is equal to the capacity of the edge e_k: $\boldsymbol{c} = (c(e_1), c(e_2), \ldots, c(e_m))$. Similarly we introduce the vector of the cost $\boldsymbol{\delta} = (\delta(e_1), \delta(e_2), \ldots, \delta(e_m))$, the vector of the demand $\boldsymbol{d} = (d(v_1), d(v_2), \ldots, d(v_n))$ and the vector of the flow $\boldsymbol{u} = (u(e_1), u(e_2), \ldots, u(e_m))$. Then we can formulate the minimum cost flow problem as the LP problem:

$$\begin{array}{rl} \text{minimize} & {}^t\boldsymbol{\delta}\boldsymbol{u} \\ \text{subject to} & Q\boldsymbol{u} = \boldsymbol{d} \\ & \boldsymbol{0} \le \boldsymbol{u} \le \boldsymbol{c},\ \boldsymbol{u} \in \mathbb{R}^m_{\ge 0}\ . \end{array} \tag{4.1}$$

Next, we formulate the shortest path problem as LP problem from view point of the minimum cost flow problem. We specify the source $s = v_1$ and the sink $t = v_m$ in the network $\mathcal{N} = (G, c, \delta, d)$ associated with the minimum cost flow problem. Moreover we define capacities and demands as follows:

(i) Capacities $c_k = 1$ for all $e_k \in E$.

(ii) Demands $d_s = -1$, $d_t = 1$ and $d_i = 0$ for all $v_i \in V \setminus \{s, t\}$.

Under these setting, we find a flow $\boldsymbol{u}$ with minimal cost $\sum_{e_k \in E} \delta_k u_k$ subject to the capacity constraint (i) and the flow conservation law (ii) as follows:

(i) $0 \leq u_k \leq c_k = 1$ for all $k = 1, \ldots, m$.

(ii) $\sum_{\partial^+(e_k)=v_i} u_k - \sum_{\partial^-(e_k)=v_i} u_k = d_i$ for all $v_i \in V$.

The shortest path problem is formulated as LP problem as follows:

$$\begin{array}{rl} \text{minimize} & {}^t\boldsymbol{\delta}\boldsymbol{u} \\ \text{subject to} & Q\boldsymbol{u} = \boldsymbol{d} \\ & \boldsymbol{0} \leq \boldsymbol{u} \leq \boldsymbol{1},\ \boldsymbol{u} \in \mathbb{R}^m_{\geq 0}\ . \end{array} \tag{4.2}$$

Here, the vector $\boldsymbol{d} = {}^t(-1, 0, \ldots, 0, 1) \in \mathbb{R}^m$ and the vector $\boldsymbol{1} = {}^t(1, 1, \ldots, 1) \in \mathbb{R}^n$. Since all entries of the constants in (4.2) are integers, all entries u_k of the feasible solution $\boldsymbol{u}$ are also integers by the Integral Flow Theorem [20]. So the value of u_k subject to the capacity constraint is equal to 0 or 1 for all $k = 1, \ldots, m$. Then, we can construct the sequence of the edges $P = (e_{i_1}, \ldots, e_{i_l})$ by picking out the edge e_{i_k} corresponding to the entry $u_{i_k} = 1$ of the feasible solution $\boldsymbol{u}$ in the digraph G. By the demand condition, the sequence P contain at least one egde e_i with $\partial^-(e_i) = s$ and at least one egde e_j with $\partial^+(e_j) = s$. The optimal solution $\boldsymbol{u}'$ of (4.2) is the solution whose cost is minimal in the feasible solutions $\boldsymbol{u}$. So the sequece of edges P' which is derived from the optimal solution $\boldsymbol{u}'$ does not contain the circuit, and it become the incidence vector of the s-t shortest path.

4.2 Maximum Flow Problem and Its Dual

In this section, we clarify the duality between flows and cutsets in the maximum flow problem through the LP problem.

4.2.1 Maximum Flow Problem as LP

Let $\mathcal{N} = (G, c, s, t)$ be a network associated with the maximum flow problem on the digraph G with n vertices and m edges. The source s and the sink t are specified vertices in V and each edge $e \in E$ is endowed with the capacity $c(e)$. In order to formulate the maximum flow problem as LP problem, we define a digraph $\overline{G} = (V, \overline{E})$ with $\overline{E} = E \cup \{e_0\}$ and $e_0 = (t, s)$. We assume that the capacity of the edge e_0 is a high value enough and u_0 is the flow on e_0. Then we can consider that a flow ${}^t({}^t\boldsymbol{u}, u_0)$ on the network $\mathcal{N}$ satisfies the flow conservation law. The maximum flow problem is reduced as follows:

$$\begin{array}{rl} \text{maximize} & {}^t\boldsymbol{\tau}' \begin{pmatrix} \boldsymbol{u} \\ u_0 \end{pmatrix} \\ \text{subject to} & \overline{Q} \begin{pmatrix} \boldsymbol{u} \\ u_0 \end{pmatrix} = \mathbf{0} \\ & \mathbf{0} \leq \boldsymbol{u} \leq \boldsymbol{c} \ . \end{array} \tag{4.3}$$

Here, the matrix $\overline{Q}$ is the incidence matrix of the digraph $\overline{G}$ and coefficients $\boldsymbol{\tau}' = (\tau_1', \ldots, \tau_m', \tau_0')$ are given by

$$\tau_i' = \begin{cases} 1 & i = 0 \\ 0 & \text{otherwise} \ . \end{cases}$$

We rewrite this formulation (4.3) to the standard form as (3.1) by introducing slack variables $\boldsymbol{v} = \boldsymbol{c} - \boldsymbol{u}$:

$$\begin{array}{rl} \text{maximize} & \left({}^t\boldsymbol{\tau}', {}^t\mathbf{0}\right) \begin{pmatrix} \boldsymbol{u} \\ u_0 \\ \boldsymbol{v} \end{pmatrix} \\ \text{subject to} & \begin{pmatrix} \overline{Q} & O \\ I_{m_0} & I_m \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ u_0 \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{c} \end{pmatrix} \\ & {}^t({}^t\boldsymbol{u}, u_0, {}^t\boldsymbol{v}) \geq \mathbf{0} \ . \end{array} \tag{4.4}$$

Here, O is the $n \times m$ zero-matrix, I_m is the $m \times m$ identity matrix and I_{m_0} is the $m \times (m+1)$ matrix defined as

$$[I_{m_0}]_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \ . \end{cases}$$

4.2.2 Dual of Maximum Flow Problem

We have the dual problem of the maximum flow problem which is formulated as LP problem (4.4) as follows:

$$\begin{array}{ll} \text{minimize} & ({}^t\mathbf{0}, {}^t\boldsymbol{c}) \cdot \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \\ \text{subject to} & \begin{pmatrix} {}^t\overline{Q} & {}^tI_{m_0} \\ O & I_m \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \geq \begin{pmatrix} \boldsymbol{\tau}' \\ \mathbf{0} \end{pmatrix} \\ & \boldsymbol{x} \in \mathbb{R}^n \ , \ \boldsymbol{y} \in \mathbb{R}^m \ . \end{array} \tag{4.5}$$

Computational experiment shows that the dual problem (4.5) computes a binary vector as the optimal solution, which presents the min-cut and min-cutset [28]. The vector $\boldsymbol{y}$ becomes the incidence vector of the minimal cutset and the vector $\boldsymbol{x}$ becomes the incidence vector of the cut which is determined by the minimal cutset in the optimal solution of (4.5).

5 Maximum Flow Problem with Gröbner Basis

5.1 Gröbner Basis

The theory of the Gröbner basis has been developed and has fruitful applications. For example, [6, 7, 16, 17, 22]. Some definitions and theorems which play an important role in our study are shown in this section.

5.1.1 Polynomial Rings and Ideaals

In this subsection, we state some of the basic definitions and notations on polynomial ideal theory that are necessary for describing the Gröbner basis, which is one of the main theme of the present thesis.

For a natural number n, a monomial in the collection of variables $x_1, \ldots, x_n$ is a product $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with non-negative integers $\alpha_1, \ldots, \alpha_n$. Introducing a vector of exponents $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ which is called a multi-index, we denote $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. If K is any field, a polynomial in $x_1, \ldots, x_n$ is a finite linear combination of monomials with coefficients in K, that is, a polynomial f in $x_1, \ldots, x_n$ with coefficients in K has the following form:

$$f = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \ .$$

Here, $c_{\boldsymbol{\alpha}} \in K$ for each multi-index $\boldsymbol{\alpha}$. We denote by $K[\boldsymbol{x}] = K[x_1, \ldots, x_n]$ the set of all polynomials in $x_1, \ldots, x_n$ with coefficients in K. $K[\boldsymbol{x}]$ becomes a commutative ring with respect to the usual addition and the multiplication.

Definition 5.1. A non-empty subset $I \subset K[x_1, \ldots, x_n]$ is called an ideal of the ring $K[x_1, \ldots, x_n]$ if it satisfies the following conditions:

(1) For $f, g \in I$, we have $f + g \in I$.

(2) For $f \in I$ and $p \in K[x_1, \ldots, x_n]$, we have $pf \in I$.

Definition 5.2. Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$. We define a subset $\langle f_1, \ldots, f_s\rangle \subset K[x_1, \ldots, x_n]$ by

$$\langle f_1, \ldots, f_s\rangle = \{p_1 f_1 + \cdots + p_s f_s | p_1, \ldots, p_s \in K[\boldsymbol{x}]\}.$$

It follows that $\langle f_1, \ldots, f_s\rangle$ becomes an ideal of $K[x_1, \ldots, x_n]$, which is called the ideal (finitely) generated by polynomials $f_1, \ldots, f_s$. Further, the celebrated Hilbert Basis Theorem asserts that an arbitrary ideal of $K[x_1, \ldots, x_n]$ is generated by a finite number of polynomials.

5.1.2 Monomial Orders

In the previouce subsection 5.1.1, we discussed the notion of polynomial rings in one and several variables. A polynomial is expressed as the sum of the monomials with coefficients. In this subsection, we introduce some monomial orders that determine how we arrange monomials in the polynomial.

Definition 5.3 (Monomial Order).
A monomial order in $K[x_1, \ldots, x_n]$ is a binary relation $\succ$ on the set of monomials $M(\boldsymbol{x}) = \{\boldsymbol{x}^{\boldsymbol{\alpha}} | \alpha \in \mathbb{Z}^n_{\geq 0}\} \subset K[x_1, \ldots, x_n]$ that satisfies the following three conditions:

(1) $\succ$ is a total (linear) ordering relation, that is, for every pair of monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{x}^{\boldsymbol{\beta}}$, exactly one of the three statements $\boldsymbol{x}^{\boldsymbol{\alpha}} \succ \boldsymbol{x}^{\boldsymbol{\beta}}$, $\boldsymbol{x}^{\boldsymbol{\alpha}} = \boldsymbol{x}^{\boldsymbol{\beta}}$, $\boldsymbol{x}^{\boldsymbol{\alpha}} \prec \boldsymbol{x}^{\boldsymbol{\beta}}$ holds.

(2) $\succ$ is compatible with the multiplication in $K[x_1, \cdots, x_n]$, in the sense that if $\boldsymbol{x}^{\boldsymbol{\alpha}} \succ \boldsymbol{x}^{\boldsymbol{\beta}}$ and $\boldsymbol{x}^{\boldsymbol{\gamma}}$ is any monomial, then we have $\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\gamma}} = \boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\gamma}} \succ \boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}} = \boldsymbol{x}^{\boldsymbol{\beta}}\boldsymbol{x}^{\boldsymbol{\gamma}}$.

(3) $\succ$ is a well-ordering, that is, every non-empty collection of monomials has the smallest element with respect to the order $\succ$.

Since a monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ is one to one correspondence with its exponents $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, a monomial order is identified with the total well-ordering $\succ$ on the set $\mathbb{Z}^n_{\geq 0}$ of multi-indices, compatible with the addition:

(2)' If $\boldsymbol{\alpha} > \boldsymbol{\beta}$ and any $\boldsymbol{\gamma} \in \mathbb{Z}_{\geq 0}$, we have $\boldsymbol{\alpha} + \boldsymbol{\gamma} > \boldsymbol{\beta} + \boldsymbol{\gamma}$.

We give some examples of monomial orders as follows.

Definition 5.4 (Lexicographic Order)**.**
Let $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{x}^{\boldsymbol{\beta}}$ be monomials in $K[x_1, \cdots, x_n]$. We say $\boldsymbol{x}^{\boldsymbol{\alpha}} \succ_{lex} \boldsymbol{x}^{\boldsymbol{\beta}}$ if in the difference $\boldsymbol{\alpha} - \boldsymbol{\beta} \in \mathbb{Z}^n$, the left-most non-zero entry is positive.

Definition 5.5 (Graded Lexicographic Order)**.**
Let $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{x}^{\boldsymbol{\beta}}$ be monomials in $K[x_1, \ldots, x_n]$. We say $\boldsymbol{x}^{\boldsymbol{\alpha}} \succ_{grlex} \boldsymbol{x}^{\boldsymbol{\beta}}$ if $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$, or, if $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and $\boldsymbol{x}^{\boldsymbol{\alpha}} \succ_{lex} \boldsymbol{x}^{\boldsymbol{\beta}}$.

Definition 5.6 (Graded Reverse Lexicographic Order)**.**
Let $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{x}^{\boldsymbol{\beta}}$ be monomials in $K[x_1, \ldots, x_n]$. We say $\boldsymbol{x}^{\boldsymbol{\alpha}} \succ_{grevlex} \boldsymbol{x}^{\boldsymbol{\beta}}$ if $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$, or, if $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and in the difference $\boldsymbol{\alpha} - \boldsymbol{\beta} \in \mathbb{Z}^n$, the right-most non-zero entry is negative.

Definition 5.7 (Weight Order)**.**
Let $\boldsymbol{w} = (w_1, \ldots, w_n)$ be a vector in $\mathbb{R}^n_{\geq 0}$, and let $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{x}^{\boldsymbol{\beta}}$ be monomials in $K[x_1, \ldots, x_n]$. Monomial order $\succ_{\boldsymbol{w}}$ is called a weight order with respect to $\boldsymbol{w}$ when the following statements hold.

$$(*) \qquad \text{If} \quad \boldsymbol{w} \cdot \boldsymbol{\alpha} > \boldsymbol{w} \cdot \boldsymbol{\beta} \qquad \text{then} \quad \boldsymbol{x}^{\boldsymbol{\alpha}} \succ \boldsymbol{x}^{\boldsymbol{\beta}} \ .$$

Here, "$\cdot$" is the inner product of vectors. In addition to the condition $(*)$, we impose the following condition:

$$(**) \qquad \text{If} \quad \boldsymbol{w} \cdot \boldsymbol{\alpha} = \boldsymbol{w} \cdot \boldsymbol{\beta} \qquad \text{then} \quad \boldsymbol{x}^{\boldsymbol{\alpha}} \succ' \boldsymbol{x}^{\boldsymbol{\beta}}$$

with a suitable monomial order $\succ'$.

For example, graded lexicographic order is one of the weight order with respect to $\boldsymbol{w} = (1, \cdots, 1)$ and provided that lexicographic order is taken as $\succ'$.

Definition 5.8 (Block Order)**.**
Let $\succ_1$ be a monomial order on $K[x_1, \ldots, x_n]$, and $\succ_2$ be a monomial order on $K[y_1, \ldots, y_m]$. Moreover, let $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{x}^{\boldsymbol{\beta}}$ be monomials in $K[x_1, \ldots, x_n]$, and let $\boldsymbol{y}^{\boldsymbol{\alpha}'}$ and $\boldsymbol{y}^{\boldsymbol{\beta}'}$ be monomials in $K[y_1, \ldots, y_m]$. Define the monomial order $\succ = (\succ_1, \succ_2)$ on $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ such that $\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\alpha}'} \succ \boldsymbol{x}^{\boldsymbol{\beta}}\boldsymbol{y}^{\boldsymbol{\beta}'}$ if the one of the following case (i) or (ii) hold:

(i) $\boldsymbol{x}^{\boldsymbol{\alpha}} \succ_1 \boldsymbol{x}^{\boldsymbol{\beta}}$.

(ii) $\boldsymbol{x}^{\boldsymbol{\alpha}} = \boldsymbol{x}^{\boldsymbol{\beta}}$ and $\boldsymbol{y}^{\boldsymbol{\alpha}'} \succ_2 \boldsymbol{y}^{\boldsymbol{\beta}'}$.

The order $\succ = (\succ_1, \succ_2)$ become a monomial order and called the block order.

Monomial orders define the leading term of the polynomial as in the following definition (4).

Definition 5.9. Let $\succ$ be a monomial order on $K[x_1, \ldots, x_n]$ and $f = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$ be a non-zero polynomial in $K[x_1, \ldots, x_n]$.
(1) The multidegree of f is defined as

$$\mathrm{multideg}(f) = \max\{\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0} : c_{\boldsymbol{\alpha}} \neq 0\},$$

where the maximum is taken with respect to $\succ$.
(2) The leading coefficient of f is defined as

$$\mathrm{LC}(f) = c_{\mathrm{multideg}(f)} \in K \ .$$

(3) The leading monomial of f is defined as

$$\mathrm{LM}(f) = x^{\mathrm{multideg}(f)} \ .$$

(4) The leading term of f is defined as

$$\mathrm{LT}(f) = \mathrm{LC}(f) \cdot \mathrm{LM}(f) \ .$$

5.1.3 Gröbner Bases

Definition 5.10 (Gröbner Bases)**.**
Fix a monomial order $\succ$ on $K[x_1, \ldots, x_n]$, and let $I \subset K[x_1, \ldots, x_n]$ be an ideal. A Gröbner basis of I with respect to $\succ$ is a finite collection of polynomials $\mathcal{G} = \{g_1, \ldots, g_t\} \subset I$ such that for every non-zero $f \in I$ it follows that $\mathrm{LT}(f)$ is divisible by $\mathrm{LT}(g_i)$ for some i.

Definition 5.11 (Reduced Gröbner Bases)**.**
A reduced Gröbner basis of $I \subset K[x_1, \ldots, x_n]$ is a Gröbner basis $\mathcal{G}$ of I such that for all distinct $p, q \in \mathcal{G}$, no monomial appearing in p is a maultiple of $\mathrm{LT}(q)$.

The Gröbner basis is a generator of the ideal in the sense that the remainder or the normal form of a polynomial with respect to the Gröbner base is uniquely determined. The precise description of the statement is given in the following proposition.

Proposition 5.12 ([6])**.** Let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a Gröbner basis with respect to a certain term order of an ideal $I \subset K[x_1, \ldots, x_n]$ and $f \in K[x_1, \ldots, x_n]$. Assume that they exist a decomposition of f such that $f = g + r$ and that the following statements (1) and (2) hold:

(1) $g \in I$.

(2) No term in r is divisible by any of $\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_s)$.

Then the decomposition is uniquely determined.

The polynomial r in the proposition 5.12 can be regarded as the remainder on division of f by $\mathcal{G}$ and it is usually called the normal form of f with respect to the Grönbner base $\mathcal{G}$ and written as $r = \overline{f}^{\mathcal{G}}$.

Corollary 5.13 ([6])**.** Let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a Gröbner basis for an ideal $I \subset K[x_1, \ldots, x_n]$ and let $f \in K[x_1, \ldots, x_n]$. Then $f \in I$ if and only if $\overline{f}^{\mathcal{G}} = 0$.

5.2 Lattices and Toric Ideals

5.2.1 Lattices and Lattice Bases

To define a toric ideal, we need the notion of a lattice.

Definition 5.14. A non-empty subset $L \subset \mathbb{Z}^n$ is called an integral lattice if it satisfies the following conditions:

(1) For $\boldsymbol{a}, \boldsymbol{b} \in L$, we have $\boldsymbol{a} + \boldsymbol{b} \in L$.

(2) For $\boldsymbol{a} \in L$ and $\lambda \in \mathbb{Z}$, we have $\lambda\boldsymbol{a} \in L$.

A finite subset $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d\} \subset L$ is called a lattice basis of L, if an arbitrary element $\boldsymbol{u} \in L$ is expressed uniquely in a linear combination as $\boldsymbol{u} = \lambda_1\boldsymbol{a}_1 + \cdots + \lambda_d\boldsymbol{a}_d$ with $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$. It is known that every integral lattices have lattice basis; although the lattice basis are not unique for the given integral lattice, the number of elements in the lattice basis is uniquely determined. Let $A = (a_{ij}) \in \mathbb{Z}^{m \times n}$ be an $m \times n$ matrix with entries in $\mathbb{Z}$. We define the $\mathbb{Z}$-kernel $\mathrm{Ker}_{\mathbb{Z}}(A)$ of A by

$$\mathrm{Ker}_{\mathbb{Z}}(A) = \{\boldsymbol{u} \in \mathbb{Z}^n | A\boldsymbol{u} = \boldsymbol{0}\}.$$

$\mathrm{Ker}_{\mathbb{Z}}(A)$ gives a class of integral lattices which plays an important part in main result. For $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, we define the support of $\boldsymbol{u}$ by $\mathrm{supp}(\boldsymbol{u}) = \{i \mid u_i \neq 0\}$.

Definition 5.15 (Elementary Vectors).
Let L be an integral lattice in $\mathbb{Z}^n$. Then $\boldsymbol{u} \in L$ is called an elementary vector of the lattice L if it satisfies the following conditions:

(1) $\mathrm{supp}(\boldsymbol{u})$ is minimal with respect to the inclusion.

(2) Non-zero elements of $\boldsymbol{u}$ are relatively prime.

5.2.2 Toric Ideals

We note that an arbitrary element $\boldsymbol{u} \in \mathbb{Z}^n$ can be written uniquely as $\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$, where $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ both have non-negative entries and have disjoint supports.

Definition 5.16 (Toric Ideals).
Let $A \in \mathbb{Z}^{m \times n}$ be an $m \times n$ matrix with entries in $\mathbb{Z}$. The toric ideal I_A associated with A is defined by the ideal generated by all binomials of the form $\boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-}$ for $\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^- \in \mathrm{Ker}_{\mathbb{Z}}(A)$ such that

$$I_A = \langle \boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} \mid \boldsymbol{u} \in \mathrm{Ker}_{\mathbb{Z}}(A) \rangle \subset K[x_1, \ldots, x_n] \ .$$

5.3 Lawrence Lifting and Universal Gröbner Bases

5.3.1 Elementary Binomials, Graver Bases and Universal Gröbner Bases

Definition 5.17 (Elemantry Binomials).
Let I_A be a toric ideal associated with $A \in \mathbb{Z}^{m \times n}$. The binomial $\boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} \in I_A$ is called an elementary binomial of the ideal I_A, if $\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$ is an elementary vector in $\mathrm{Ker}_{\mathbb{Z}}(A)$. The set of all elementary binomials of I_A is denoted by $\mathcal{E}_{\mathcal{A}}$.

Definition 5.18 (Universal Gröbner Bases).
The union of all reduced Gröbner bases of I_A with respect to every term orders is called the universal Gröbner basis of I_A and denoted by $\mathcal{U}_A$.

There exist the infinite number of term orders. It is proved, however, that number of term orders that have different Gröbner basis for the given ideal are finite, the universal Gröbner basis of an ideal becomes a finite set.

Theorem 5.19 ([22]). Let I_A be a toric ideal. It follows that

$$\mathcal{E}_A \subseteq \mathcal{U}_A \ .$$

Definition 5.20 (Unimodular Matrices).
An $m \times n$ matrix $A \in \mathbb{Z}^{m\times n}$ with rank $d(\leq m, n)$ is called unimodular if all non-zero $d \times d$ minors of A have the same absolute value.

Theorem 5.21 ([22]). Let A be a unimodular integer matrix. It follows that

$$\mathcal{E}_A = \mathcal{U}_A \ .$$

5.3.2 Lawrence Lifting

Definition 5.22. Let $A \in \mathbb{Z}^{m\times n}$ be a $m \times n$ matrix. Define an $(m+n)\times 2n$ enlarged matrix $\Lambda(A)$ of the following form

$$\Lambda(A) = \begin{pmatrix} A & O \\ I_n & I_n \end{pmatrix} \ .$$

Here, I_n is the $n \times n$ identity matrix and O is the $m \times n$ zero-matrix. The matrix $\Lambda(A)$ is called the Lawrence lifting of A.

It follows from Definition 5.22 that

$$\mathrm{Ker}_{\mathbb{Z}}(\Lambda(A)) = \{{}^t({}^t\boldsymbol{u}, -{}^t\boldsymbol{u})|\boldsymbol{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)\} \ .$$

Then it can be seen that the $\mathrm{Ker}_{\mathbb{Z}}(A)$ and $\mathrm{Ker}_{\mathbb{Z}}(\Lambda(A))$ are isomorphic integer lattices. Note that the toric ideal I_A and $I_{\Lambda(A)}$ associated with the Lawrence lifting are quite different. The toric ideal $I_{\Lambda(A)}$ is of the form:

$$I_{\Lambda(A)} = \langle \boldsymbol{x}^{\boldsymbol{u}^+}\boldsymbol{y}^{\boldsymbol{u}^-} - \boldsymbol{x}^{\boldsymbol{u}^-}\boldsymbol{y}^{\boldsymbol{u}^+} | \boldsymbol{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)\rangle \subset K[\boldsymbol{x}, \boldsymbol{y}] \ .$$

5.4 Maximum Flow Problem with Gröbner Bases

5.4.1 Conti-Traverso's Algorithm for IP

We consider the case where all coefficients of the standard form of the IP problem (3.2) are non-negative integers, i.e., $A\boldsymbol{u} = \boldsymbol{b}$ with $A \in \mathbb{Z}_{\geq}^{m\times n}$, $\boldsymbol{u} \in \mathbb{Z}_{\geq}^{n}$ and $\boldsymbol{b} \in \mathbb{Z}_{\geq}^{m}$. We introduce indeterminates $z_1, \ldots, z_m$ for each $A\boldsymbol{u} = \boldsymbol{b}$ and exponentiate to obtain an equality

$$z_i^{a_{i1}u_1+\cdots+a_{in}u_n} = z_i^{b_i} \quad (i = 1, \ldots, m)$$

Multiplying the left and right hand sides of these equation, and rearranging the exponents, we get equality as follows:

$$\prod_{j=1}^{n}(\prod_{i=1}^{m} z_i^{a_{ij}})^{u_j} = \prod_{i=1}^{m} z_i^{b_i} \ .$$

Then we express monomials $f_j = \prod_{i=1}^m z_i^{a_{ij}}$. We introduce indeterminates $x_1, \ldots, x_n$ and consider the ideal

$$I = \langle f_1 - x_1, \ldots, f_n - x_n \rangle \subset K[z_1, \ldots, z_m, x_1, \ldots, x_n] \; .$$

In order to solve the IP problem by using the Gröbner basis, we have to compute the Gröbner basis $\mathcal{G}$ of the above ideal I with respect to the monomial order which is adapted to the IP problem. We define the monomial order adapted to the IP problem as follows.

Definition 5.23 (Monomial Order Adapted to IP)**.**
An adapted oarder of the LP problem $\succ_{ip}$ on $K[z_1, \ldots, z_m, x_1, \ldots, x_n]$ is obtained by the following procedure:

(i) Introduce a suitable monomial order $\succ_1$ in $K[z_1, \ldots, z_m]$.

(ii) Introduce the weight order $\succ_{\boldsymbol{w}}$ in $K[x_1, \ldots, x_n]$ using $\boldsymbol{w}$ as the weight.

(iii) Define $\succ_{ip}$ in $K[z_1, \ldots, z_m, x_1, \ldots, x_n]$ as the block order $\succ_{ip} = (\succ_1, \succ_{\boldsymbol{w}})$.

In order to solve the IP problems, we have to express $f = z_1^{b_1} \cdots z_m^{b_m}$ as the monomial of $f_1, \ldots, f_n$. Since the adapted order is the elimination order with respect to the variables $z_1, \ldots, z_m$, the IP problem has a feasible solution if and only if the normal form $\overline{f}^{\mathcal{G}}$ is a monomial in $K[x_1, \ldots, x_n]$. If $\overline{f}^{\mathcal{G}}$ is the monomial in $K[x_1, \ldots, x_n]$, then it follows immediately from Definition 5.23 (ii) of the adapted order, the exponent of the normal form gives an optimal solution for the IP problem. Consequently we have the algorithm for solving IP problems with all $a_{ij}, b_i \geq 0$.

Algorithm 5.24 (Conti-Traverso).

$$\begin{aligned}
\textbf{Input} &: A, \boldsymbol{b}, \boldsymbol{w}, \text{an adapted monomial order } \succ_{ip} \\
\textbf{Output} &: \text{an optimal solution of } \boldsymbol{u},\ \text{as long as it exists} \\
\textbf{Step1} &: f_j := \prod_{i=1}^{m} z_i^{a_{ij}}, I := \langle f_1 - x_1, \dots, f_n - x_n \rangle \\
&\quad \textbf{Calculate}\ \ \mathcal{G} := \text{Gröbner basis of } I \text{ with respect to } \succ_{ip} \\
\textbf{Step2} &: f := \prod_{i=1}^{m} z_i^{b_i} \\
&\quad \textbf{Calculate}\ \ g := \overline{f}^{\mathcal{G}} \\
\textbf{Step3} &: \textbf{IF}\ \ g \in K[x_1, \dots, x_n]\ \ \textbf{THEN} \\
&\qquad \text{its exponent vector gives a solution} \\
&\quad \textbf{ELSE}\ \ \text{there is no solution}
\end{aligned}$$

Next we show a toric version of the above algorithm.

Algorithm 5.25 (Conti-Traverso Toric Version).

$$\begin{aligned}
\textbf{Input} &: A, \boldsymbol{b}, \boldsymbol{w}, \text{an adapted monomial order } \succ_{ip} \\
\textbf{Output} &: \text{an optimal solution of } \boldsymbol{u}^*,\ \text{as long as it exists} \\
\textbf{Step1} &: I_A := \langle \boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} | \boldsymbol{u} \in \mathrm{Ker}_{\mathbb{Z}}(A) \rangle \\
&\quad \textbf{Calculate}\ \ \mathcal{G} := \text{Gröbner basis of } I_A \text{ with respect to } \succ_{ip} \\
\textbf{Step2} &: \text{one of the feasible solutions } \boldsymbol{u}_0 \\
&\quad \textbf{Calculate}\ \ \overline{\boldsymbol{x}^{\boldsymbol{u}_0}}^{\mathcal{G}} = \boldsymbol{x}^{\boldsymbol{u}^*} \\
&\qquad \text{its exponent vector } \boldsymbol{u}^* \text{ gives a solution}
\end{aligned}$$

5.4.2 Maximum Flow Problem as IP

In this subsection, we discuss the maximum flow problem by using the Conti-Traverso algoritm. We will give a slightly diferent formulation of the maximum flow problem from (4.4) in order to apply the Conti-Traverso algoritm. Let $\mathcal{N} = (G, c, s, t)$ be a network associated with the maximum flow problem on the digraph G with n vertices and m edges. The source s and the sink t are specifyed vertices and each edge e is endowed with the capacity $c(e)$. In what follows, we consider the following assumptions on G:

(A1) G is connected and has no loops.

(A2) G has no (s,t) edges and has no edges whose tail is t or head is s.

Expressing $\widetilde{Q} = \widetilde{Q}^+ - \widetilde{Q}^-$ with $(0,1)$ matrices $\widetilde{Q}^+$ and $\widetilde{Q}^-$, introducing slack variables $\boldsymbol{v} = \boldsymbol{c} - \boldsymbol{u}$ and rewriting the objective function from maximization to minimization, we can formulate the maximum flow problem as the IP problem as follows:

$$\begin{array}{rl} \text{minimize} & {}^t\boldsymbol{\tau}\boldsymbol{v} \\ \text{subject to} & \begin{pmatrix} \widetilde{Q}^+ & \widetilde{Q}^- \\ I_n & I_n \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \widetilde{Q}^- \boldsymbol{c} \\ \boldsymbol{c} \end{pmatrix} \\ & {}^t({}^t\boldsymbol{u}, {}^t\boldsymbol{v}) \geq {}^t\boldsymbol{0} \ . \end{array} \tag{5.1}$$

Here, I_n is the $n \times n$ identity matrix. We denote by A the matrix coefficients appeared in the constraint, i.e.,

$$A = \begin{pmatrix} \widetilde{Q}^+ & \widetilde{Q}^- \\ I_n & I_n \end{pmatrix} .$$

By this formulation, it can be solved by using the Gröbner basis technique of toric ideals through Conti-Traverso's algorithm.

5.4.3 Generators of Toric Ideals

For the general toric ideal I_M associated with a matrix $M \in \mathbb{Z}^{m \times n}$, it is known that a set of generators of I_M is computed by using Hosten-Sturmfels' algorithm [18]. The following proposition presents one of the special case to compute the set of generators of I_M without using Hosten-Sturmfels' algorithm.

Proposition 5.26 ([23]). Let $M \in \mathbb{Z}^{m \times n}$ be an integer matrix. If there exists a lattice vector $\boldsymbol{v} \in \mathrm{Ker}_{\mathbb{Z}}(M)$ whose all entries are positive, then the toric ideal I_M is generated by the binomials corresponding to a lattice basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ of the $\mathrm{Ker}_{\mathbb{Z}}(M)$. Thus we have

$$I_M = \langle \boldsymbol{x}^{\boldsymbol{v}_i^+} - \boldsymbol{x}^{\boldsymbol{v}_i^-} \mid i = 1, 2, \ldots, k \rangle \ .$$

In subsection 5.4.5, we will give the characterization of a set of generators of the toric ideal $I_{\widetilde{Q}}$ associated with the reduced incidence matrix $\widetilde{Q}$ of the digraph G.

5.4.4 Circuit and Cocircuit

We consider a digraph $G = (V, E)$ with n vertices and m edges. For a circuit C in G, we denote by C^+ the set of forward edges and denote by C^- the set of backward edges of C. A minimal set of edges $C^* \subset E$ that makes the graph G disconnected is called a cocircuit. So, the cocircuit C^* divides the graph G into the two connected components G^+ and G^-. The cocircuit C^* is divided into the disjoint union $C^* = C^{*+} \sqcup C^{*-}$: edges in C^{*+} have the tail in G^+ and the head in G^-; the edges in C^{*-} have the tail in G^- and the head in G^+.

Definition 5.27.
(1) For a circuit $C \subset E$ of a digraph $G = (V, E)$ with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$, we define the incidence vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$ of C by

$$\gamma_k = \begin{cases} +1 & \text{if} \quad e_k \in C^+ \\ -1 & \text{if} \quad e_k \in C^- \\ 0 & \text{if} \quad e_k \notin C \end{cases}$$

A circuit C is directed if and only if all entries of the incidence vector $\boldsymbol{\gamma}$ of the circuit C are non-negative.
(2) We define the incidence vector $\boldsymbol{\gamma}^*$ of a cocircuit $C^* \subset E$ in the same way as the definition of the incidence vector of a circuit.
(3) The circuit space L of G is defined as the subspace of $\mathbb{Q}^m$ generated by incidence vectors of the circuits of G. The cocircuit space $L^\perp$ of G is defined as the subspace of $\mathbb{Q}^m$ generated by incidence vectors of the cocircuits of G.

The following proposition show that the circuit space L and the cocircuit space $L^\perp$ are orthgonal and complemental.

Proposition 5.28 ([2])**.** Let $G = (V, E)$ be a digraph. The cocircuit space $L^\perp$ is the orthogonal complement of the circuit space L. More precisely, the circuit space and the cocircuit space can be characterized as $L = \mathrm{Ker}(Q)$ and $L^\perp = \mathrm{Im}({}^tQ)$, respectively by the incidence matrix Q.

Let $M \subset \mathbb{Q}^m$ be a subspace. A nonzero vector $\boldsymbol{x} \in M$ is called an elementary vector of M if its support is minimal with respect to the inclusion, i.e., the vectors with strictly smaller support are not contained in M.

Proposition 5.29 ([2])**.** Let $L, L^\perp \subset \mathbb{Q}^m$ denote the circuit and the cocircuit space of G respectively. Then a vector $\boldsymbol{x} \in \mathbb{Q}^m$ is an elementary vector of L if and only if it is a scalar multiple of an incidence vector of a circuit.

Also, elementary vectors of the cocircuit space $L^{\perp}$ can be characterized as scalar multiples of of incidence vectors of cocircuits.

It follows the above proposition that the toric ideal associated with the Lawrence lifting of the incidence matrix of a graph is minimally generated by binomials corresponding to all incidence vectors of the circuits [3].

5.4.5 Universal Gröbner Bases Associated with the Maximum Flow Problam

We consider the maximum flow problem on the network $\mathcal{N} = (G, c, s, t)$. We assume (A1) and (A2) in Sebsection 5.4.2. So we allow the existence of parallel and anti-parallel edges. In 5.4.2, we formulate the maximum flow problem as the IP problem as follows:

$$\begin{array}{ll} \text{minimize} & {}^t\boldsymbol{\tau}\boldsymbol{v} \\ \text{subject to} & \begin{pmatrix} \widetilde{Q}^{+} & \widetilde{Q}^{-} \\ I_n & I_n \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \widetilde{Q}^{-}\boldsymbol{c} \\ \boldsymbol{c} \end{pmatrix} \end{array} \tag{5.2}$$

It follows from the definition that the toric ideal I_A associated with the matrix coefficients A of (5.2) becomes

$$I_A = \langle \boldsymbol{x}^{\boldsymbol{u}^+}\boldsymbol{y}^{\boldsymbol{u}^-} - \boldsymbol{x}^{\boldsymbol{u}^-}\boldsymbol{y}^{\boldsymbol{u}^+} | \boldsymbol{u} \in \mathrm{Ker}_{\mathbb{Z}}(\widetilde{Q}) \rangle \ ,$$

which is proved to be identical with the toric ideal associated with the Lawrence lifting

$$\Lambda(\widetilde{Q}) = \begin{pmatrix} \widetilde{Q} & \mathbf{0} \\ I_n & I_n \end{pmatrix}$$

of $\widetilde{Q}$. It is well known that the incidence matrix Q of a graph is totally unimodular, so the matrix $\widetilde{Q}$ is also totally unimodular. Further, by a straightforward computation, we can prove that the Lawrence type matrix $\Lambda(\widetilde{Q})$ is totally unimodular. This implies that Theorem 5.21 can be applied to the ideal $I_A = I_{\Lambda(\widetilde{Q})}$. Thus we have

$$\mathcal{E}_A = \mathcal{U}_A \ .$$

It can be seen that the computation of the universal Gröbner basis $\mathcal{U}_A$ of I_A is to determin the set of all elementary binomials $\mathcal{E}_A$ of I_A. Thus we are ready to prove the main theorem.

Theorem 5.30 ([26])**.** *Let I_A be the toric ideal associated with the matrix coefficients A of (5.2). Then the universal Gröbner basis of I_A consists of binomials corresponding to all incidence vectors of circuits and s-t paths of G.*

Proof. A key idea of the proof is to construct a digraph $\widehat{G} = (\widehat{V}, \widehat{E})$ from the digraph $G = (V, E)$ by the following procedure: Let $V = \{v_1(= s), v_2, \dots, v_m(= t)\}$ be the set of vertices of G. We make a new vertex $\hat{v}_1$ by identifying the source s with the sink t in V. We define the new vertex set $\widehat{V}$ by $\widehat{V} = \{\hat{v}_1, v_2, \dots, v_{m-1}\}$. Next, we define the new edge set $\widehat{E} = \{\hat{e}_1, \dots, \hat{e}_n\}$ by the following incidence relations:

$$\partial^-(\hat{e}_i) = \begin{cases} \hat{v}_1 & \text{if} \quad \partial^-(e_i) = s \\ v_j & \text{if} \quad \partial^-(e_i) = v_j \ (v_j \neq s) \end{cases} ,$$

$$\partial^+(\hat{e}_i) = \begin{cases} \hat{v}_1 & \text{if} \quad \partial^+(e_i) = t \\ v_j & \text{if} \quad \partial^+(e_i) = v_j \ (v_j \neq t) \end{cases} ,$$

where $E = \{e_1, e_2, \dots, e_n\}$ denotes the edge set of G. The simple example above illustrates the above construction.

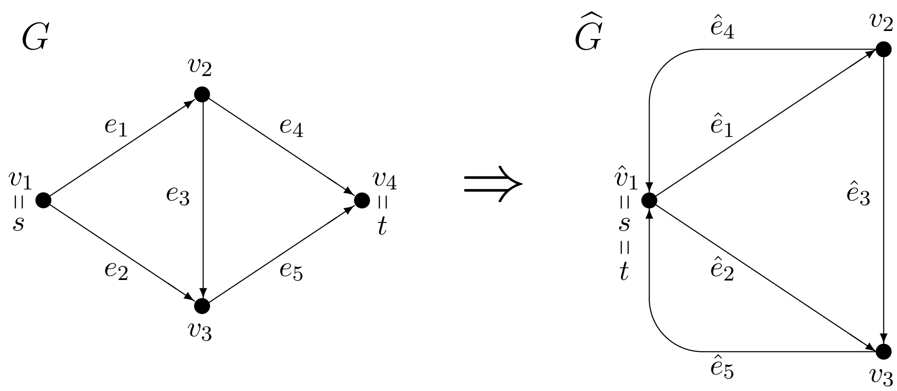

Figure 1: the new digraph

It follows from the assumption that the digraph G has no (s,t) edges and no loops that the new digraph $\widehat{G}$ has no loops. We compute the incidence matrix $\widehat{Q}$ of $\widehat{G}$: Let $\boldsymbol{q}_i$ be a i^{th} row vector of the incidence matrix Q. We note that the first row vector $\hat{\boldsymbol{q}}_1$ of $\widehat{Q}$ is of the form $\hat{\boldsymbol{q}}_1 = \boldsymbol{q}_1 + \boldsymbol{q}_m$. This shows that the incidence matrix $\widehat{Q}$ is an $(m-1) \times n$ matrix consisting of the row vectors $\hat{\boldsymbol{q}}_1, \boldsymbol{q}_2, \dots, \boldsymbol{q}_{m-1}$. On the other hand, $\widetilde{Q}$ consists of the row vectors $\boldsymbol{q}_2, \dots, \boldsymbol{q}_{m-1}$. Since the sum of all row vectors of Q is zero: $\boldsymbol{q}_1 + \boldsymbol{q}_2 + \cdots + \boldsymbol{q}_m = \boldsymbol{0}$, we have $\hat{\boldsymbol{q}}_1 = \boldsymbol{q}_1 + \boldsymbol{q}_m = -(\boldsymbol{q}_2 + \boldsymbol{q}_3 + \cdots + \boldsymbol{q}_{m-1})$, which means that the first row vector $\hat{\boldsymbol{q}}_1$ of $\widehat{Q}$ is a linear combination of the other row vectors $\boldsymbol{q}_2, \dots, \boldsymbol{q}_{m-1}$. Thus we have

$$\mathrm{Ker}_{\mathbb{Z}}(\widetilde{Q}) = \mathrm{Ker}_{\mathbb{Z}}(\widehat{Q}) .$$

Since the matrix $\widehat{Q}$ is the incidence matrix of the digraph $\widehat{G}$, it follows from Proposition 5.29 that an elementary vector of $\mathrm{Ker}(\widehat{Q})$ is a scalar multiple of an incidence vector of the circuit of $\widehat{G}$. So the computation of the set of all elementary binomials $\mathcal{E}_A$ is reduced to the enumeration of circuits of $\widehat{G}$. It is easy to see that the set of circuits of $\widehat{G}$ yields the set of circuits and the set of s-t paths of G. This completes the proof of the theorem. □

Weismantel [27] shows that the universal Gröbner basis associated with the minimum cost flow problem can be characterized by the set of all circuits of the graph.

Applying Theorem 5.30, we determine the generators of the toric ideal associated with the reduced incidence matrix $\widetilde{Q}$ of the digraph G under some assumptions for G. In order to get the desired result, we make the additional assumption:

(A3) There exists at least one s-t directed path through an arbitrary edge $e \in E$ in the digraph $G = (V, E)$.

Then we will prove the following proposition under the assumptions (A1),(A2) and (A3).

Proposition 5.31 ([26]). Let G be a digraph which satisfies the assumptions (A1), (A2) and (A3). Then the integral lattice of $\mathrm{Ker}_{\mathbb{Z}}(\widetilde{Q})$ is generated by the incidence vectors of the s-t directed paths and the directed circuits in the digraph G.

Proof. It follows from Theorem 5.30 that the integral lattice $\mathrm{Ker}_{\mathbb{Z}}(\widetilde{Q})$ is generated by the incidence vectors of the s-t paths and the circuits in a digraph G. So it is enough to prove that all incidence vectors of s-t paths or circuits of G are expressed as integral linear combinations of the incidence vectors of s-t directed paths and directed circuits of G. We will prove that the incidence vectors of arbitrary s-t paths are expressed as integral linear combinations of the incidence vectors of s-t directed paths and directed circuits. Let $P = (e_1, \ldots, e_k)$ be an arbitrary s-t path in the digraph G. Let $e_i \in P$ be the first backward edge in the path P. We express the path P as $P = (\mathcal{P}_1, e_i, P_2)$ with the directed path $\mathcal{P}_1 = (e_1, \cdots, e_{i-1})$ and the path $P_2 = (e_{i+1}, \ldots, e_k)$. By assumption (A3), there exists at least one s-t directed path $\mathcal{P}'$ through the edge e_i. We express the directed path $\mathcal{P}'$ as $\mathcal{P}' = (\mathcal{P}'_1, e_i, \mathcal{P}'_2)$ in terms of the directed paths $\mathcal{P}'_1$ and $\mathcal{P}'_2$. Noting that the edge $e_i \in P$ and the edge $e_i \in \mathcal{P}'$ have the opposite direction, we have a s-t walk $W = (\mathcal{P}'_1, P_2)$ and a s-t directed walk $\mathcal{W} = (\mathcal{P}_1, \mathcal{P}'_2)$. We see that the s-t walk W has less backward edges than the s-t path P does. Applying the same procedure as above to the s-t walk W, it is easy to verify that the sum of the incidence vectors of the s-t path P and the s-t directed path $\mathcal{P}'$ is written as the sum of the incidence vectors of the s-t directed walks and the s-t directed paths. In order to prove the assertion of the proposition for the s-t paths, it is enough to prove the following lemma.

Lemma 5.32 ([26])**.** Let $\mathcal{W}$ be a directed walk. Then the incidence vector of $\mathcal{W}$ can be written as the sum of the incidence vector of one directed path and those of suitable directed circuits.

Proof. Let $\mathcal{W} = (e_1, \ldots, e_k)$ be a directed walk and let $e_i \in \mathcal{W}$ be the first repeated edge: $\mathcal{W} = (e_1, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e'_{i-1}, e_i, e'_{i+1}, \ldots, e_k)$. Then the incidence vector of the directed walk $\mathcal{W}$ is written as the sum of the incidence vector of the directed walk $\mathcal{W}' = (e_1, \ldots, e_{i-1}, e_i, e'_{i+1}, \ldots, e_k)$ and that of the directed circuit $\mathcal{C} = (e_i, e_{i+1}, \ldots, e'_{i-1})$. We see that the directed walk $\mathcal{W}'$ has less repeated edge than the directed walk $\mathcal{W}$ does. Applying the same procedure as above to the directed walk $\mathcal{W}'$, the incidence vector of $\mathcal{W}$ can be written as the sum of the incidence vector of one directed trail and those of the suitable directed circuits. It is easy to verify that the incidence vector of the directed trail is expressed as the sum of the incidence vector of one directed path and those of suitable directed circuits. This completes our proof of the lemma. $\square$

Next we prove the assertion for the circuits. Let $C = (e_1, \ldots, e_\ell)$ be a circuit in G and let $e_i \in C$ be one of the backward edge. By assumption (A3), there exists at least one s-t directed path $\mathcal{P}$ through the edge e_i. We express the directed path $\mathcal{P}$ as $\mathcal{P} = (\mathcal{P}_1, e_i, \mathcal{P}_2)$ with the directed paths $\mathcal{P}_1$ and $\mathcal{P}_2$. We set $C' = C \setminus \{e_i\}$. Since the edge $e_i \in C$ and the edge $e_i \in \mathcal{P}$ have the opposite direction, we see that $W = (\mathcal{P}_1, C', \mathcal{P}_2)$, become the s-t walk, which is not necessary directed. Then we can apply the similar arguments in the first part of the proof of the proposition, we can express the sum of the incidence vector of C and the incidence vector of $\mathcal{P}$ as the sum of the incidence vectors of s-t directed paths and the incidence vectors of directed circuits. We have completed the proof of the proposition. $\square$

By using Proposition 5.31 and Proposition 5.26, we can determine a set of generators of $I_{\widetilde{Q}}$ as follows:

Theorem 5.33 ([26])**.** Let $\widetilde{Q}$ be the reduced incidence matrix of a digraph G which satisfies the assumptions (A1), (A2) and (A3), and let $\mathcal{P}$ and $\mathcal{C}$ be a set of all incidence vectors of the s-t directed paths and a set of all incidence vectors of the directed circuits, respectively. Then we have

$$I_{\widetilde{Q}} = \langle \boldsymbol{x}^{\boldsymbol{p}} - 1, \boldsymbol{x}^{\boldsymbol{\gamma}} - 1 \mid \boldsymbol{p} \in \mathcal{P}, \boldsymbol{\gamma} \in \mathcal{C} \rangle \ .$$

Proof. It follows from Proposition 5.31 that the integral lattice $\mathrm{Ker}_{\mathbb{Z}}(\widetilde{Q})$ is generated by the incidence vectors of the s-t directed paths and the incidence vectors of the directed circuits in a digraph G. We consider the sum $\boldsymbol{v}$ of all incidence vectors in $\mathcal{P}$. Since we assume that there exist at least one s-t directed path through an arbitrary edge e_i in the digraph G, we see that all entries of the vector $\boldsymbol{v}$ is more than 1. So we prove that the vector $\boldsymbol{v}$ satisfies the assumption of Proposition 5.26. Hence we see that the toric ideal $I_{\widetilde{Q}}$ is generated by binomials corresponding to all incidence vectors of s-t directed paths and all incidence vectors of directed circuits. □

In the maximum flow problem, we usually consider the network on the digraph without directed circuits. In this case, we have the following corollary:

Corollary 5.34 ([21, 26])**.** Let $\widetilde{Q}$ be the reduced incidence matrix of a digraph G without directed circuits. We also assume that the digraph G satisfies the assumptions (A1), (A2) and (A3). Then we have

$$I_{\widetilde{Q}} = \langle \boldsymbol{x}^{\boldsymbol{p}} - 1 \mid \boldsymbol{p} \in \mathcal{P} \rangle$$

where $\mathcal{P}$ is a set of all incidence vectors of the s-t directed paths.

6 Eigenvalue Problem over Min-Plus Algebra

In this section, we focus on the eigenvalue problem of matrices with entries in min-plus algebra. The eigenvalue of the min-plus algebra gives a characterization of the network.

6.1 Min-Plus Algebra

6.1.1 Basic Notations and Definitions

Let $\mathbb{R}$ be the field of real numbers. We define the min-plus algebra $\mathbb{R}_{\min}$ by $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$, with the binary operations $\oplus$ and $\otimes$ such as

$$a \oplus b = \min\{a, b\} \quad , \quad a \otimes b = a + b \ .$$

Both of them are associative and commutative such as

$$\begin{array}{rcl} a \oplus (b \oplus c) = (a \oplus b) \oplus c & , & a \otimes (b \otimes c) = (a \otimes b) \otimes c \ , \\ a \oplus b = b \oplus a & , & a \otimes b = b \otimes a \ . \end{array}$$

for all $a, b, c \in \mathbb{R}_{\min}$. The algebra $\mathbb{R}_{\min}$ has the identity $\varepsilon = +\infty$ with respect to $\oplus$ such as

$$a \oplus \varepsilon = \varepsilon \oplus a = \min\{a, +\infty\} = a \ ,$$

and the identity $e = 0$ with respect to $\otimes$ such as

$$a \otimes e = e \otimes a = a + 0 = a \ .$$

If $x \neq \varepsilon$, there exists the unique inverse y $(= -x)$ of x with respect to $\otimes$ such as

$$x \otimes y = e \ .$$

The operation $\otimes$ is distributive with respect to $\oplus$ such as

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z) \ .$$

The identity $\varepsilon = +\infty$ with respect to $\oplus$ is absorbing for $\otimes$ such as

$$x \otimes \varepsilon = \varepsilon \otimes x = x + \infty = +\infty = \varepsilon \ .$$

The operation $\oplus$ is idempotent such as

$$x \oplus x = \min\{x, x\} = x \ .$$

Definition 6.1. For $x \in \mathbb{R}_{\min}$ and $k \in \mathbb{N}$, the k^{th} power of x is defined by

$$x^{\otimes k} = \underbrace{x \otimes x \otimes \ldots \otimes x}_{k \ \text{times}} \ .$$

In $\mathbb{R}_{\min}$, the k^{th} power of x reduces to the conventional multiplication $x^{\otimes k} = kx$.

It is easy to verify that the min-plus power has the following properties for $x, y \in \mathbb{R}_{\min}$ and $m, n \in \mathbb{N}$:

(1) $x^{\otimes m} \otimes x^{\otimes n} = x^{\otimes (m \otimes n)}$;

(2) $(x^{\otimes m})^{\otimes n} = x^{\otimes (m^{\otimes n})}$;

(3) $x^{\otimes 1} = x$;

(4) $x^{\otimes m} \otimes y^{\otimes m} = (x \otimes y)^{\otimes m}$.

6.1.2 Matrix Algebra over Min-Plus Algebra

For positive integers $m, n \in \mathbb{N}$, we denote by $\mathbb{R}_{\min}^{m\times n}$ the set of all $m \times n$ matrices with entries in $\mathbb{R}_{\min}$. We define the several operations in $\mathbb{R}_{\min}^{m\times n}$ analogous to those in the conventional matrix algebra as follows.

Definition 6.2.
(1) For $A = (a_{ij}) \in \mathbb{R}_{\min}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}_{\min}^{m\times n}$, we define their sum $A \oplus B \in \mathbb{R}_{\min}^{m\times n}$ by

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \min\{a_{ij}, b_{ij}\} \ .$$

(2) For $A = (a_{ij}) \in \mathbb{R}_{\min}^{m\times k}$ and $B = (b_{ij}) \in \mathbb{R}_{\min}^{k\times n}$, we define their product $A \otimes B \in \mathbb{R}_{\min}^{m\times n}$ by

$$[A \otimes B]_{ij} = \bigoplus_{\ell=1}^{k} (a_{i\ell} \otimes b_{\ell j}) = \min_{\ell=1,2,\ldots,k} \{a_{i\ell} + b_{\ell j}\} \ .$$

(3) For $A = (a_{ij}) \in \mathbb{R}_{\min}^{m\times n}$, we define the transpose ${}^tA \in \mathbb{R}_{\min}^{n\times m}$ of A by

$$[{}^tA]_{ij} = a_{ji} \ .$$

(4) Define the matrix $I_n \in \mathbb{R}_{\min}^{n\times n}$ by

$$[I_n]_{ij} = \begin{cases} e \text{ if } i = j \\ \varepsilon \text{ if } i \neq j \ . \end{cases}$$

Then it can be seen that $A \otimes I_n = I_n \otimes A = A$ for $A \in \mathbb{R}_{\min}^{n\times n}$, which means that $I = I_n$ becomes the identity with respect to the matrix multiplication in $\mathbb{R}_{\min}^{n\times n}$.
(5) For $A \in \mathbb{R}_{\min}^{n\times n}$ and $k \in \mathbb{N}$, we define the k^{th} power of A by

$$A^{\otimes k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}} \ .$$

We set $A^{\otimes 0} = I$, for $k = 0$.
(6) For $A = (a_{ij}) \in \mathbb{R}_{\min}^{m\times n}$ and $\alpha \in \mathbb{R}_{\min}$, we define the scalar multiplication $\alpha \otimes A \in \mathbb{R}_{\min}^{m\times n}$ by

$$[\alpha \otimes A]_{ij} = \alpha \otimes a_{ij} \ .$$

The operation $\oplus$ is commutative in $\mathbb{R}_{\min}^{m\times n}$, but not $\otimes$ is. The operation $\otimes$ is distributive with respect to the operation $\oplus$ in the matrix algebra $\mathbb{R}_{\min}^{m\times n}$. Also $\oplus$ is idempotent in $\mathbb{R}_{\min}^{m\times n}$, that is, we have $A \oplus A = A$.

6.2 Networks and Min-Plus Algebra

6.2.1 Networks on Graphs

Let $G = (V, E)$ be a digraph with n vertices and m edges. We assign a real number $w(e)$ to each edge $e \in E$; $w(e)$ is called the weight of the edge e. In this section, the pair $\mathcal{N} = (G, w)$ is called a network on the digraph G.

Definition 6.3. Let $\mathcal{N} = (G, w)$ be a network on the digraph G. We define the weighted adjacency matrix $B(G) = (b_{ij}) \in \mathbb{R}^{n\times n}$ of $\mathcal{N}$ by

$$b_{ij} = \begin{cases} w((v_i, v_j)) & \text{if} \quad (v_i, v_j) \in E \\ 0 & \text{if} \quad (v_i, v_j) \notin E \end{cases}$$

Definition 6.4. For the circuit C, let $\ell(C)$ be the length of C and let $\sigma(C)$ be the weight of C. Then we define the average weight of the circuit C by $\dfrac{\sigma(C)}{\ell(C)}$.

Definition 6.5. Let $\mathcal{N}$ be a network on the digraph $G = (V, E)$ with n vertices. We denote by b^*_{ij} $(i, j = 1, \ldots, n)$ the minimal value of weights of all v_i-v_j paths in G. We set $b^*_{ij} = \infty$, if there exists no v_i-v_j path. Define the minimal weight matrix $B^*(G) = B^* \in \mathbb{R}^{n\times n}_{\min}$ by $B^* = (b^*_{ij})$.

6.2.2 Adjacency Matrices with Values in Min-Plus Algebra

Let $G = (V, E)$ be a digraph with n vertices and m edges, and let $\mathcal{N} = (G, w)$ be a network on G. Moreover, let $B(G) = (b_{ij})$ be the weighted adjacency matrix of the network $\mathcal{N}$.

Definition 6.6. We define the weighted adjacency matrix $\widetilde{B} = (\widetilde{b}_{ij})$ with values in $\mathbb{R}_{\min}$ of the network $\mathcal{N}$ by

$$\widetilde{b}_{ij} = \begin{cases} b_{ij} & \text{if} \quad (v_i, v_j) \in E \\ +\infty & \text{if} \quad (v_i, v_j) \notin E \end{cases}$$

Since the addition $+$ in $\mathbb{R}$ means the operation $\otimes$ in $\mathbb{R}_{\min}$, we have the weight $\sigma(P)$ of the path $P = (v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_s}, v_{i_s})$ in the digraph G as follows:

$$\sigma(P) = \bigotimes_{k=0}^{s-1} \widetilde{b}_{i_k i_{k+1}} \ .$$

For a matrix $A \in \mathbb{R}^{n\times n}_{\min}$, we can define the network $\mathcal{N} = (G, w)$ on the digraph G whose weighted adjacency matrix with values in $\mathbb{R}_{\min}$ coincides with A. We denote such network by $\mathcal{N}(A)$ and call the network associated with the matrix $A \in \mathbb{R}^{n\times n}_{\min}$.

Proposition 6.7 ([15, 30])**.** Given a matrix $A \in \mathbb{R}_{\min}^{n\times n}$. Assume that the network $\mathcal{N}(A)$ has no circuits of negative weight. Then the minimal weight matrix A^* of the network $\mathcal{N}(A)$ can be obtained by the following power sum computation:

$$A^* = I \oplus A \oplus A^{\otimes 2} \oplus \cdots \ .$$

6.3 Eigenvalue Problem over Min-Plus Algebra

In this section, we show that min-plus eigenvalues and eigenvectors admit a graph theoretical interpretation. Throughout this section, we consider the digraph $G = (V, E)$ with the set of n vertices $V = \{1, 2, \ldots, n\}$ and m edges. Then, an edge $e \in E$ can be expressed as a pair $e = (i, j),\ i, j \in V$.

Definition 6.8. Given a matrix $A \in \mathbb{R}_{\min}^{n\times n}$, we say that $\lambda \in \mathbb{R}_{\min}$ is a right eigenvalue of A if there exists $\boldsymbol{x} \in \mathbb{R}_{\min}^{n}$ such that $\boldsymbol{x} \neq {}^t(\varepsilon, \varepsilon, \ldots, \varepsilon)$ and

$$A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x} \ .$$

The vector $\boldsymbol{x}$ is called the right eigenvector of A belonging to the right eigenvalue λ. Similarly, we say that $\lambda' \in \mathbb{R}_{\min}$ is a left eigenvalue of A if there exists $\boldsymbol{y} \in \mathbb{R}_{\min}^{n}$ such that $\boldsymbol{y} \neq {}^t(\varepsilon, \varepsilon, \ldots, \varepsilon)$ and

$${}^tA \otimes \boldsymbol{y} = \lambda' \otimes \boldsymbol{y} \quad (\text{ or } {}^t\boldsymbol{y} \otimes A = \lambda' \otimes {}^t\boldsymbol{y}) \ .$$

The vector $\boldsymbol{y}$ is called the left eigenvector of A belonging to the left eigenvalue λ'.

We allow a right eigenvalue and a left eigenvalue to have the value ε. First, we characterize matrices A having the right or the left eigenvalue ε. A matrix $A \in \mathbb{R}_{\min}^{n\times n}$ is said to have ε-columns if it has at least one columm whose all entries are ε. Similarly, if a matrix A has at least one row whose all entries are ε then it is said to have ε-rows.

Proposition 6.9. The identity ε of $\oplus$ is a right eigenvalue of A if and only if A has ε-columns. Similarly, ε is a left eigenvalue of A if and only if A has ε-rows.

Proof. We prove the assertion for the right eigenvalues. Let $\boldsymbol{x}$ be a right eigenvector of A belonging to the right eigenvalue $\lambda = \varepsilon$. From the definition, the right eigenvector $\boldsymbol{x}$ has at least one entry $x_j \neq \varepsilon$. Then we will prove that all entries of the j^{th} column of A are equal to ε. Suppose that one entry a_{ij} of the j^{th} column of A satisfies $a_{ij} \neq \varepsilon$ for some i, then we have $a_{ij} \otimes x_j \neq \varepsilon$. On the other hand, we have $\lambda \otimes x_i = \varepsilon$ since $\lambda = \varepsilon$, which lead to the contradiction. Thus we have proved that the all entries of the j^{th} column of A are ε. This completes the proof of the if part. Next we prove the only if part. We assume that all entries of j^{th} column of A are ε. Then it is easy to show that the vector $\boldsymbol{x} = {}^t(x_1, \ldots, x_j, \ldots, x_n)$ with $x_j = a \neq \varepsilon$ and $x_i = \varepsilon$ $(i \neq j)$ becomes a right eigenvector of A belonging to the right eigenvalue ε. We have proved the only if part. If we note that the expression ${}^t\boldsymbol{y} \otimes A = \lambda' \otimes {}^t\boldsymbol{y}$ is equivalent to the expression ${}^tA \otimes \boldsymbol{y} = \lambda' \otimes \boldsymbol{y}$, we can easily prove the assertion for the left eigenvalue λ' from the assertion for the right eigenvalues. Thus we have completed the proof of the proposition. □

Next, we characterize matrices $A \in \mathbb{R}_{\min}^{n \times n}$ having the right eigenvalue $\lambda \neq \varepsilon$. We consider the case where matrices A do not have ε-columns.

Lemma 6.10. Assume that $A \in \mathbb{R}_{\min}^{n \times n}$ does not have ε-columns. Then the network $\mathcal{N}(A)$ associated with the matrix A has at least one circuit.

Proof. The assumption is equivalent to the fact that any vertices in the network $\mathcal{N}(A)$ become the head of at least one edge. That is, for any $v_1 \in V$, there exist at least one edge e_1 with $e_1 = (v_2, v_1), v_2 \in V$. Applying the same procedure, we can find a sequence of vertices, $v_1, \ldots, v_i, v_{i+1}, \ldots$ such that $e_i = (v_{i+1}, v_i) \in E$ and in the sequence we find a circuit C since the number of vertices is finite. □

Definition 6.11. Let $\lambda(\neq \varepsilon)$ be an element of $\mathbb{R}_{\min}$. We define a matrix A_λ by $[A_\lambda]_{ij} = [A]_{ij} - \lambda$.

We assume that the matrix $A \in \mathbb{R}_{\min}^{n \times n}$ does not have ε-columns. Then it follows from Lemma 6.10 that the network $\mathcal{N}(A)$ associated with the matrix A has circuits. Let λ be the minimal value of the average weight of circuits in $\mathcal{N}(A)$ and consider the network $\mathcal{N}(A_\lambda)$ associated with the matrix A_λ. Since the network $\mathcal{N}(A_\lambda)$ does not have circuits with negative weights, we can compute by Proposition 6.7 the minimal weight matrix A_λ^* by the power sum : $A_\lambda^* = I \oplus A_\lambda \oplus A_\lambda^{\otimes 2} \oplus \cdots$.

Theorem 6.12. Let $A \in \mathbb{R}_{\min}^{n \times n}$ be a matrix without ε-columns and let $\lambda \neq \varepsilon$ be the minimal average weight of circuits in the network $\mathcal{N}(A)$. Let $C = ((v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1))$ be the circuit in $\mathcal{N}(A)$ expressed as a sequence of edges and having the minimal average weight λ.
Then the column vectors $[A_\lambda^*]_{v_1}, \ldots, [A_\lambda^*]_{v_k}$ of the minimal weight matrix A_λ^* of the network $\mathcal{N}(A_\lambda)$ become the right eigenvectors of A belonging to the right eigenvalue λ.

Proof. We represent by ν one of the vertices in $\{v_1, v_2, \ldots, v_k\}$. Then it is enough to prove the following equality:

$$A \otimes [A_\lambda^*]_\nu = \lambda \otimes [A_\lambda^*]_\nu \tag{1}$$

First, we compute the left hand side of the equality (1). From the definition of A_λ, we have $A = \lambda \otimes A_\lambda$ and if we use the natation: $A_\lambda^+ = A_\lambda \oplus A_\lambda^{\otimes 2} \oplus \cdots$, we have

$$\begin{aligned} A \otimes [A_\lambda^*]_\nu &= \lambda \otimes A_\lambda \otimes [A_\lambda^*]_\nu \\ &= \lambda \otimes A_\lambda \otimes [I \oplus A_\lambda \oplus A_\lambda^{\otimes 2} \oplus \cdots]_\nu \\ &= \lambda \otimes [A_\lambda \oplus A_\lambda^{\otimes 2} \oplus \cdots]_\nu \\ &= \lambda \otimes [A_\lambda^+]_\nu \end{aligned}$$

where I is the identity matrix in $\mathbb{R}_{\min}^{n\times n}$. Thus the equality (1) is rewritten as:

$$\lambda \otimes [A_\lambda^+]_\nu = \lambda \otimes [A_\lambda^*]_\nu \ .$$

So it is enough to prove $[A_\lambda^+]_\nu = [A_\lambda^*]_\nu$. The entries $[A_\lambda^*]_{i\nu}$ $(i = 1, 2, \ldots, n)$ are given by:

$$[A_\lambda^*]_{i\nu} = \begin{cases} \varepsilon \oplus [A_\lambda^+]_{i\nu} & \text{if} \quad i \neq \nu \\ e \oplus [A_\lambda^+]_{i\nu} & \text{if} \quad i = \nu \end{cases}$$

So the identity for the $i \neq \nu$ case is trivial. Consider the identity for $i = \nu$ case. The entries $[A_\lambda^+]_{\nu\nu}$ indicate the minimal weight of ν-ν path in the network $\mathcal{N}(A_\lambda)$. Since λ is the minimal average weight of the circuit C in $\mathcal{N}(A)$ and ν is an arbitrary vertex in the circuit C, the minimal weight of ν-ν path is equal to e. Thus we have proved $[A_\lambda^+]_\nu = [A_\lambda^*]_\nu$. This completes the proof of the theorem. □

Let $A \in \mathbb{R}_{\min}^{n\times n}$ be a matrix without ε-columns. Then we have proved that A has an right eigenvalue $\lambda \neq \varepsilon$ which is the minimal average weight of the circuits in the network $\mathcal{N}(A)$ associated with A. Next we will prove that this right eigenvalue is the only right eigenvalue of the matrix A.

Proposition 6.13 ([25])**.** If the matrix $A \in \mathbb{R}_{\min}^{n\times n}$ has an right eigenvalue $\lambda \neq \varepsilon$, there exists a circuit in the network $\mathcal{N}(A)$ whose average weight is equal to λ.

Proof. Let $\lambda \neq \varepsilon$ be the right eigenvalue of A. By the definition, a right eigenvector $\boldsymbol{x}$ belonging to the right eigenvalue λ has at least one entry $x_{v_1} \neq$

ε. Hence we have $[A \otimes \boldsymbol{x}]_{v_1} = \lambda \otimes x_{v_1} \neq \varepsilon$. Thus we can find a vertex v_2 with $a_{v_1 v_2} \otimes x_{v_2} = \lambda \otimes x_{v_1}$. This implies that $a_{v_1 v_2} \neq \varepsilon$, $x_{v_2} \neq \varepsilon$ and $(v_1, v_2) \in E$. By the same argument we can find $v_3 \in V$ with $a_{v_2 v_3} \otimes x_{v_3} = \lambda \otimes x_{v_2}$ and $(v_2, v_3) \in E$. Applying the same procedure, we find the sequence of vertices $v_1, v_2, \dots, v_i, \dots$ such that $(v_{i-1}, v_i) \in E$. Since the number of vertices is finite, we can find the subsequence $(v_h, v_{h+1}, \dots, v_{h+k})$, in which the vertices are pairwise distinct except $v_h = v_{h+k}$. Then the sequence of edges

$$C = ((v_h, v_{h+1}), (v_{h+1}, v_{h+2}), \dots (v_{h+k-1}, v_h))$$

express the circuit C. The circuit C has the length $\ell(C) = k$ and the weight $\sigma(C) = \bigotimes_{j=0}^{k-1} a_{v_{h+j} v_{h+j+1}}$, where $v_h = v_{h+k}$. By the construction of sequence of vertices, we have

$$\bigotimes_{j=0}^{k-1} (a_{v_{h+j} v_{h+j+1}} \otimes x_{v_{h+j+1}}) = \lambda^{\otimes k} \otimes \bigotimes_{j=0}^{k-1} x_{v_{h+j}}.$$

Converting $\otimes$ to $+$ in conventional algebra, we have

$$\sum_{j=0}^{k-1} (a_{v_{h+j} v_{h+j+1}} + x_{v_{h+j+1}}) = k\lambda + \sum_{j=0}^{k-1} x_{v_{h+j}}.$$

Using the fact that

$$\sum_{j=0}^{k-1} x_{v_{h+j+1}} = \sum_{j=0}^{k-1} x_{v_{h+j}},$$

we obtain

$$\bigotimes_{j=0}^{k-1} a_{v_{h+j} v_{h+j+1}} = k\lambda,$$

which means that $\sigma(C) = k\lambda$. Therefore we have proved that the average weight of the circuit C is

$$\frac{\sigma(C)}{\ell(C)} = \frac{k\lambda}{k} = \lambda$$

This completes the proof of the proposition. $\square$

Proposition 6.13 shows that an arbitrary right eigenvalue of a matrix A comes from the average weight of circuits in $\mathcal{N}(A)$.

Theorem 6.14. Assume that $A \in \mathbb{R}_{\min}^{n \times n}$ does not have ε-columns. The matrix A has the unique right eigenvalue λ which is equal to the minimal average weight of circuits in $\mathcal{N}(A)$ associated with the matrix A.

Proof. It follows from the assumption for A and Theorem 6.12 that A has an eigenvalue λ which is not equal to ε. Let $\boldsymbol{x} = {}^t(x_1, \ldots, x_n)$ be an eigenvector belonging to the eigenvalue λ. We see from Proposition 6.13 that the network $\mathcal{N}(A)$ contains at least one circuit. and let $C = ((v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1))$ be any one of circuits in $\mathcal{N}(A)$. Then we have

$$\bigotimes_{j=1}^{k} a_{v_j v_{j+1}} \otimes x_{v_{j+1}} \geq \lambda^{\otimes k} \otimes \bigotimes_{j=1}^{k} x_{v_j} \quad (v_{k+1} = v_1)$$

Using the same argument as in the proof of Proposition 6.13, we have

$$\frac{\sigma(C)}{\ell(C)} \geq \frac{k\lambda}{k} = \lambda \ .$$

This inequality holds for an arbitrary right eigenvalue $\lambda \neq \varepsilon$ of A and for an arbitrary circuit C in $\mathcal{N}(A)$. So the right eigenvalue λ of A has to be smaller or equal to the minimal average weight of the circuit in $\mathcal{N}(A)$. By Proposition 6.13, any right eigenvalue comes from the average weight of a circuit in $\mathcal{N}(A)$. Therefore the right eigenvalue $\lambda \neq \varepsilon$ of A is uniquely determined and equal to the minimal average weight of the circuit in $\mathcal{N}(A)$. □

Here, we are concerned only with the right eigenvalue. We show that the unique right eigenvalue coincide with the unique left eigenvalue.

Corollary 6.15. Assume that the matrix $A \in \mathbb{R}_{\min}^{n \times n}$ does not have ε-columns and ε-rows. Then the unique right eigenvalue of A is identical with the unique left eigenvalue of A.

Proof. By the definition, λ' is the left eigenvalue of A if and only if it is the right eigenvalue of the transpose ${}^tA$ of A. Let $G = (V, E)$ be the digraph which defines the network $\mathcal{N}(A)$. We define the new digraph ${}^tG = (V, {}^tE)$ with the set of vertices V and the set of edges ${}^tE$: The set of edges ${}^tE$ is defined by $(i, j) \in {}^tE$ if and only if $(j, i) \in E$. Let w be the weight function on E defined by the matrix A. We define the weight function $\bar{w}$ on ${}^tE$ by $\bar{w}((i, j)) = w((j, i))$. Thus we can define the network $({}^tG, \bar{w})$. It is easy to verify the weighted adjacency matrix with values in $\mathbb{R}_{\min}$ of the network $({}^tG, \bar{w})$ coincides with the matrix ${}^tA$. It follows from the definition that the minimal average weight of circuits in the network (G, w) become the minimal average weight of the network $({}^tG, \bar{w})$ and vice versa. By the assumption of the corollary A and ${}^tA$ have the unique eigenvalue which is the minimal average weight of circuits in (G, w) or $({}^tG, \bar{w})$ respectively. Then we see that the unique eigenvalue of A and ${}^tA$ coincide. Thus we have completed the proof of the corollary. □

7 Conclusion

In the present thesis, in order to interpret the max flow min cutset theorem from the view point of LP duality, we adopt a special LP formulation of the maximum flow problem. Computational experiment suggests that the optimal solution to the LP dual problem of our formulation becomes binary vector, giving the minimal cut and the minimal cutset. However, we cannot make clear the reason why the optimal solution of the dual problem of the maximum flow problem becomes the binary vector, although the dual problem is the LP problem. This remains as a future subject of study.

In order to apply the Gröbner basis technique to the maximum flow problem, we give another LP formulation of the maximum flow problem, which is slightly different from the previous formulation. Based on this formulation, we give the characterization of the universal Gröbner basis associated with the maximum flow problem in terms of circuits and paths of the digraph (Theorem 5.30). Via this characterization, we have only to enumerate all circuits and s-t paths of a given graph in order to compute the universal Gröbner basis associated with the maximum flow problem. However, we do not have the efficient algorithm for such enumeration. It seems that the computational complexity of enumerating circuits and paths may be rather large but probably smaller than that of Buchberger's algorithm in the computation of the Gröbner basis. We remain the verification of the practical efficiency of our results for the future study. Even if we have a good algorithm for enumerating of circuits and paths, our approach using Conti-Traverso's algorithm will not be practically useful because there exists excellent combinatorial algorithms such as the preflow-push algorithm. Nevertheless, we think that our result has own mathematical importance. Moreover if we can interpret various combinatorial algorithms in terms of the universal Gröbner basis, we can find some significance of our result beyond the mathematical interest. Further, under a suitable assumption for the digraph, we prove Theorem 5.33 which give the characterization of the integer kernel of the reduced incidence matrix of a digraph as a consequence of Theorem 5.30. In order to prove this Theorem 5.33, we had to prove Lemma 5.32 asserting that the incidence vector of a the directed walk can be written as the sum of the incidence vector of one directed path and those of suitable directed circuits. We think that the result of Lemma 5.32 itself is the interesting and have some importance in the theory of flow-network.

Finally, we characterize the eigenvalue and corresponding eigenvectors of the matrices with entries in Min-Plus algebra in terms of the network on the digraph associated with the matrix. We show that the minimal average weight of the circuit in the network become the min-plus eigenvalue. Also we show that the corresponding eigenvectors appear as the column vectors of the minimal weight matrix (the distance matrix) of the specified network which is obtained from the network by subtracting the minimal average weight from every edges of the graph. As mentioned above, enumerating circuits and paths is the important problem in the combinatorial theory of graphs. If we have an algorithm for computing the specified circuit with minimal average weight in the network by solving Min-Plus eigenvalue problem, this will become the first step to enumerate circuits of the digraph.

8 Acknowledgments

I would like to express my sincere gratitude to my supervisor, Professor Yoshihide Watanabe at Doshisha University for providing me the happiest time and this precious study. This time and experiences are no other substitute for that. Thank you very much. I am also grateful to Professor Jyunta Matsukidaira at Ryukoku University, Professor Seiji Saitou at Doshisha University and Mr. Daisuke Ikegami for their fruitful discussions and various comments. Moreover, I thank Professor Taketomo Mitsui and Professor Yorimasa Oshime for theire carefully judging. Finally I thank my colleagues and family for their help.

Sennosuke WATANABE

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